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クォーク閉じ込め：最近の発展と今後の課題

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Quark Confinement: A review of recent developments and future problems

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Based on e-Print: arXiv:1409.1599 [hep-th] (submitted to Physics Report)
Quark confinement: dual superconductor picture based on a non-Abelian Stokes theorem and reformulations of Yang-Mills theory
v1 (277 pages including 59 figures and 13 tables)
v2: available soon.

§ Introduction

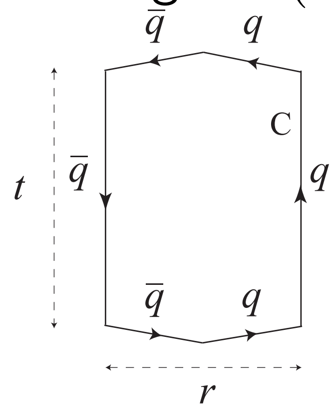
We follow the Wilson criterion [Wilson,1974]. For a closed path C, we define the **Wilson loop operator** $W_C[\mathscr{A}]$ for the **non-Abelian Yang-Mills field** $\mathscr{A}_\mu(x)$ by

$$W_C[\mathscr{A}] := \mathrm{tr}\left[\mathscr{P}\exp\left\{ig_{\mathrm{YM}}\oint_C dx^\mu \mathscr{A}_\mu(x)\right\}\right]/\mathrm{tr}(\mathbf{1}), \quad \mathscr{A}_\mu(x) = \mathscr{A}_\mu^A(x)T_A, \tag{1}$$

where $\mathscr{P}$ denotes the **path-ordering** prescription. Note that $W_C[\mathscr{A}]$ is gauge invariant. In the Yang-Mills theory, we consider the **Wilson loop average** $W(C)$, i.e., a vacuum expectation value of the Wilson loop operator $W_C[\mathscr{A}]$ for a closed loop C:

$$W(C) = \langle W_C[\mathscr{A}]\rangle_{\mathrm{YM}}. \tag{2}$$

For a rectangular loop C of side lengths t and r, the Wilson loop average $W(C)$ is related to a **static quark-antiquark potential** $V_{q\bar{q}}(r)$ as



$$W(C) \sim \exp[-tV_{q\bar{q}}(r)], \quad (t \gg r). \tag{3}$$

Therefore, $V_{q\bar{q}}(r)$ is obtained in the gauge-independent way from

$$V_{q\bar{q}}(r) = \lim_{t\to\infty}\frac{-1}{t}\ln W(C). \tag{4}$$

Simulation result of the static quark potential

Static quark-antiquark potential $V_{q\bar{q}}(r)$= Coulomb+Linear:

$$V_{q\bar{q}}(r) = -\frac{\alpha}{r} + \sigma r + c,$$

wjth the parameters, σ: string tension [mass^2], α: dimensionless [mass^0], c: [mass^1].

- $\sigma \neq 0$ confinement $V_{q\bar{q}}(r) \to \infty$ as $r \to \infty$
- $\sigma = 0$ deconfinement $V_{q\bar{q}}(r) < \infty$ as $r \to \infty$

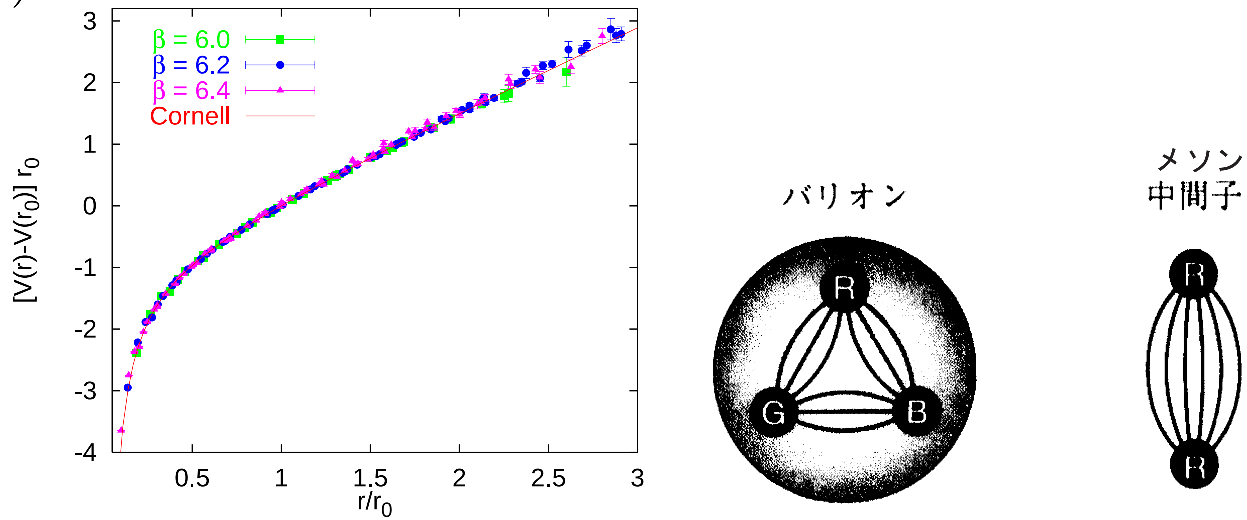


Figure 1: The static quark-antiquark potential $V(r)$ as a function of the distance r in SU(3) Yang-Mills theory, which is obtained by numerical simulations in the framework of lattice gauge theory. Note that the potential is normalized so that $V(r_0) = 0$ and $\beta = 2N/g^2_{\rm YM}$ for $SU(N)$. See G.S. Bali,[hep-ph/0001312], Phys.Rept.**343**, 1 (2001).

Dual superconductor hypothesis for quark confinement

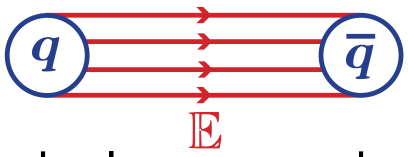
[Nambu (1974), 't Hooft (1975), Mandelstam (1976), Polyakov (1975,1977) ...]

The key ingredients for the hypothesis of dual superconductor are as follows.

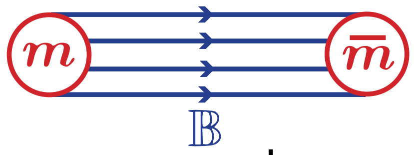
* Dual Meissner effect

In the dual superconductor, chromoelectric flux is squeezed into tubes.

[← In the ordinary superconductor, magnetic flux is squeezed into tubes]



QCD vacuum=dual superconductor ← dual →



superconductor

* condensation of chromomagnetic monopoles

The dual superconductivity is caused by condensation of chromomagnetic monopoles.

[← The ordinary superconductivity is cased by condensation of electric charge into Cooper pairs.]

In order to establish the dual superconductivity, we must answer the following questions:

* How to introduce magnetic monopoles in the Yang-Mills theory without scalar fields?

cf. 't Hooft-Polyakov magnetic monopole

* How to define the duality in the non-Abelian gauge theory?
* How to preserve the original (non-Abelian) gauge symmetry?
* How to extract the infrared dominant mode $\mathscr{V}$ for confinement?

Collective infrared gluonic degrees of freedom

For $D = 4$,

- instantons, merons (calorons):
  0-dimensional, i.e., localized lumps of quantized topological charge

- magnetic monopoles:
  1-dimensional closed world-lines of sources and sinks (of quantized chromomagnetic flux) Abelian magnetic monopole under the Abelian projection?

- center vortices:
  2-dimensional closed world-surfaces (of quantized chromomagnetic flux with a thickness related to the QCD scale) center vortices under the center projection?

For $D = 3$,

- magnetic monopoles:
  0-dimensional world-points of sources and sinks (of quantized chromomagnetic flux)

- center vortices:
  1-dimensional closed world-lines (of quantized chromomagnetic flux)

Part I:

Theoretical framework towards the dual superconductor picture for quark confinement and mass gap (a non-Abelian Stokes theorem and reformulations of the Yang-Mills theory)

For the summary of this part, see
K.-I. Kondo, S. Kato A. Shibata, and T. Shinohara, Proc. of XIth Quark Confinement and the Hadron Spectrum (QCHSXI), 8-12 September 2014, St Petersburg, Russia, arXiv:1412.8008 [hep-th]

§ Non-Abelian Stokes theorem

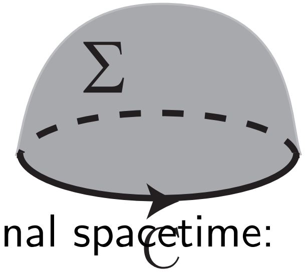
We consider how the Wilson loop can be related to the magnetic monopole.

First, we consider the Abelian case. The Abelian Wilson loop operator $W_C[A]$ for a loop C is cast into the surface integral over the surface Σ_C bounded by C using the **Stokes theorem**:

$$W_C[A] = \exp\left[ie \oint_C dx^\mu A_\mu\right] \Longrightarrow W_C[A] = \exp\left[ie \int_{\Sigma:\partial\Sigma=C} dS^{\mu\nu}(x(\sigma))F_{\mu\nu}(x(\sigma))\right].$$

Introduce an antisymmetric tensor $\Theta^{\mu\nu}_{\Sigma_C}$ (the **vorticity tensor**) with the support only on the surface Σ_C

$$\Theta^{\mu\nu}_{\Sigma}(x) := \int_{\Sigma:\partial\Sigma=C} d^2 S^{\mu\nu}(x(\sigma))\delta^D(x - x(\sigma)).$$



Then the surface integral is rewritten into the spacetime integral over the D-dimensional spacetime:

$$W_C[A] = \exp\{ie(\Theta_\Sigma, F)\}, \quad (\Theta_\Sigma, F) := \int d^D x \Theta^{\mu\nu}_{\Sigma}(x) F_{\mu\nu}(x).$$

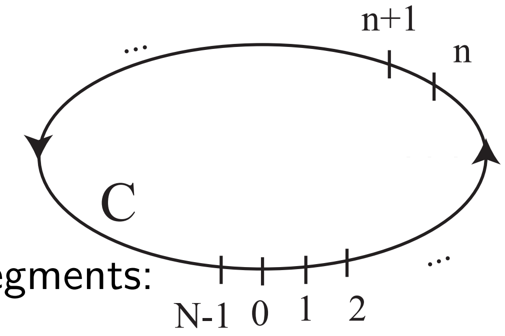
The Hodge decomposition can be used to define the electric current j and the **magnetic current** k

$$W_C[A] = \exp\{ie(N_\Sigma, j) + ie(\Xi_\Sigma, k)]\}, \quad N_\Sigma := \delta\Delta^{-1}\Theta_\Sigma, \quad \Xi_\Sigma := \delta\Delta^{-1*}\Theta_\Sigma. \qquad (1)$$

The electric current j is non-vanishing:$j := \delta F \neq 0$, while the magnetic current k is vanishing due to the Bianchi identity and there is no magnetic contribution to the Wilson loop:

$$k := \delta^* F = {}^* dF = {}^* ddA = 0 \Longrightarrow W_C[A] = \exp\{ie(N_\Sigma, j)]\}.$$

Next, we consider the non-Abelian case. The non-Abelian Wilson loop operator $W_C[\mathscr{A}]$ (in the representation R) is written using the trace and the path ordering as



$$W_C[\mathscr{A}] := \mathrm{tr}_R \left\{ \mathscr{P} \exp \left[-ig_{\mathrm{YM}} \oint_C \mathscr{A} \right] \right\} / \mathrm{tr}_R(\mathbf{1}).$$

The **path ordering** $\mathscr{P}$ is defined by dividing the path C into N infinitesimal segments:

$$W_C[\mathscr{A}] = \lim_{N\to\infty,\epsilon\to 0} \mathrm{tr}_R \left\{ \mathscr{P} \prod_{n=0}^{N-1} \exp \left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A} \right] \right\} / \mathrm{tr}_R(\mathbf{1}). \tag{3}$$

The troublesome path ordering in the non-Abelian Wilson loop operator can be removed as first shown for $G = SU(2)$ by [Diakonov and Petrov (1989)], which is the **non-Abelian Stokes theorem** (NAST).

Moreover, the NAST for the Lie group G can be obtained as the path-integral representation of the Wilson loop operator using the **coherent state of the Lie group** G in an unified way. [Kondo (1998), Kondo and Taira (2000), Kondo (2008)]. We follow the standard steps for the path integral:
1. We insert a **complete set** of states at each partition point:

$$\mathbf{1} = \int d\mu(g(x_n)) \left| g(x_n), \Lambda \right\rangle \left\langle g(x_n), \Lambda \right| \quad (n = 1, \cdots, N-1). \tag{4}$$

where $d\mu(g)$ is an invariant measure on G and the state is normalized $\langle g(x_n), \Lambda | g(x_n), \Lambda \rangle = 1$.
Here the **coherent state** $|g, \Lambda\rangle$ is constructed by operating a group element $g \in G$ to a **reference state** $|\Lambda\rangle$ (e.g., the highest-weight state) for a given representation R of the Wilson loop we consider:

$$|g, \Lambda\rangle = g \left| \Lambda \right\rangle, \quad g \in G. \tag{5}$$

2. We replace the trace of the operator $\mathscr{O}$ by the integral:

$$\mathrm{tr}_{\mathrm{R}}(\mathscr{O})/\mathrm{tr}_R(\mathbf{1}) = \int d\mu(g(x_0)) \langle g(x_0), \Lambda| \, \mathscr{O} \, |g(x_0), \Lambda\rangle \, , \tag{6}$$

3. We take the limit $N \to \infty$ and $\epsilon \to 0$ appropriately such that $N\epsilon$ is fixed:

$$W_C[\mathscr{A}] = \lim_{N\to\infty,\epsilon\to 0} \prod_{n=0}^{N-1} \int d\mu(g(x_n)) \prod_{n=0}^{N-1} \langle g(x_{n+1}), \Lambda| \exp\left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A}\right] |g(x_n), \Lambda\rangle.$$

For taking the limit $\epsilon \to 0$ in the final step, it is sufficient to retain the $O(\epsilon)$ terms.

$$\begin{aligned}
&\langle g_{n+1}, \Lambda| \exp\left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A}\right] |g_n, \Lambda\rangle \\
=&\langle \Lambda| g(x_{n+1})^\dagger \exp\left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A}\right] g(x_n)|\Lambda\rangle = \langle \Lambda| \exp\left[-ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \mathscr{A}^g\right] |\Lambda\rangle \\
=&\langle \Lambda| \left[1 - ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} d\tau \mathscr{A}^g(\tau) + O(\epsilon^2)\right] |\Lambda\rangle \\
=&1 - ig_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \langle \Lambda|\mathscr{A}^g|\Lambda\rangle + O(\epsilon^2) \quad (\langle \Lambda|\Lambda\rangle = 1) \\
=&\exp\left[-i\epsilon g_{\mathrm{YM}} \int_{x_n}^{x_{n+1}} \langle \Lambda|\mathscr{A}^g|\Lambda\rangle\right] + O(\epsilon^2).
\end{aligned} \tag{7}$$

Here $\mathscr{A}^g(x)$ agrees with the gauge transformation of $\mathscr{A}(x)$ by the group element g:

$$\mathscr{A}^g(x) := g(x)^\dagger \mathscr{A}(x) g(x) + i g_{\rm YM}^{-1} g(x)^\dagger d g(x). \tag{8}$$

Defining the one-form A^g from the Lie algebra valued one-form $\mathscr{A}^g$ by

$$A^g := \langle \Lambda | \mathscr{A}^g | \Lambda \rangle, \tag{9}$$

we arrive at a path-integral representation of the Wilson loop operator (pre-NAST):

$$W_C[\mathscr{A}] = \int [d\mu(g)]_C \exp\left(-i g_{\rm YM} \oint_C A^g \right), \quad [d\mu(g)]_C := \lim_{N\to\infty, \epsilon\to 0} \prod_{n=0}^{N-1} d\mu(g(x_n)). \tag{10}$$

The path-ordering has disappeared and $A^g(x)$ is no longer non-Abelian field.
Therefore, we can apply the (usual) Stokes theorem to obtain a **non-Abelian Stokes theorem** (NAST):

$$W_C[\mathscr{A}] = \int [d\mu(g)]_\Sigma \exp\left[-i g_{\rm YM} \int_{\Sigma:\partial\Sigma=C} F^g \right], \quad F^g = dA^g. \tag{11}$$

Here we replaced the integration measure on the loop C by the integration measure on the surface Σ:

$$[d\mu(g)]_\Sigma := \prod_{x\in\Sigma:\partial\Sigma=C} d\mu(g(x)), \tag{12}$$

by inserting additional integration measures, $1 = \int d\mu(g(x))$ for $x \in \Sigma - C$.

For the **highest-weight state** $|\Lambda\rangle = (\lambda_a)$ of a representation R of a group G, we define a matrix ρ with the matrix element ρ_{ab} by

$$\rho := |\Lambda\rangle \langle\Lambda| , \quad \rho_{ab} := |\Lambda\rangle_a \langle\Lambda|_b = \lambda_a \lambda_b^* . \tag{13}$$

Since $|\Lambda\rangle$ is normalized: $\langle\Lambda|\Lambda\rangle = \lambda_a \lambda_a^* = 1$, the trace of ρ has a unity: $\mathrm{tr}(\rho) = \rho_{aa} = 1$.Moreover, the matrix element $\langle\Lambda| \mathscr{O} |\Lambda\rangle$ of an arbitrary matrix $\mathcal{O}$ is written in the trace form:

$$\langle\Lambda| \mathscr{O} |\Lambda\rangle = \mathrm{tr}(\rho\mathscr{O}), \tag{14}$$

since $\langle\Lambda| \mathscr{O} |\Lambda\rangle = \lambda_b^* \mathscr{O}_{ba} \lambda_a = \rho_{ab} \mathscr{O}_{ba} = \mathrm{tr}(\rho\mathscr{O})$.

By using the operator ρ, the "Abelian" field A^g is written in the trace form of a matrix:

$$A^g(x) = \mathrm{tr}\{\rho \mathscr{A}^g(x)\} = \mathrm{tr}\{g(x)\rho g^\dagger(x)\mathscr{A}(x)\} + i g_{\rm YM}^{-1} \mathrm{tr}\{\rho g^\dagger(x) dg(x)\}. \tag{15}$$

By introducing the traceless field $\tilde{\boldsymbol{n}}(x)$ defined by [which we call the **color (direction) field**

$$\tilde{\boldsymbol{n}}(x) := g(x) \left[\rho - \frac{\mathbf{1}}{\mathrm{tr}(\mathbf{1})}\right] g^\dagger(x) = g(x)\rho g^\dagger(x) - \frac{\mathbf{1}}{\mathrm{tr}(\mathbf{1})}, \tag{16}$$

the "Abelian" field A^g is rewritten as

$$A_\mu^g(x) = \mathrm{tr}\{\tilde{\boldsymbol{n}}(x)\mathscr{A}_\mu(x)\} + i g_{\rm YM}^{-1} \mathrm{tr}\{\rho g^\dagger(x) \partial_\mu g(x)\}, \tag{17}$$

We proceed to perform the **decomposition of the Yang-Mills field** $\mathscr{A}_\mu(x)$ into two pieces:

$$\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x). \tag{18}$$

We simply require that $\mathscr{X}_\mu(x)$ satisfies the condition:[which we call the second **defining equation**]

$$\text{(ii)} \quad \mathscr{X}_\mu(x) \cdot \boldsymbol{n}(x) = 2\mathrm{tr}\{\mathscr{X}_\mu(x)\boldsymbol{n}(x)\} = 0. \tag{19}$$

Then $\mathscr{X}_\mu(x)$ disappears from the Wilson loop operator, since $A_\mu^g(x)$ is written without $\mathscr{X}_\mu(x)$:

$$A_\mu^g(x) = \mathrm{tr}\{\tilde{\boldsymbol{n}}(x)\mathscr{V}_\mu(x)\} + ig_{\mathrm{YM}}^{-1}\mathrm{tr}\{\rho g^\dagger(x)\partial_\mu g(x)\}. \tag{20}$$

Consequently, the Wilson loop operator $W_C[\mathscr{A}]$ can be reproduced by the **restricted field** variable $\mathscr{V}_\mu(x)$ alone. This is the **restricted field dominance** in the Wilson loop operator. For arbitrary loop C and any representation R,

$$\text{(a)} \quad W_C[\mathscr{A}] = W_C[\mathscr{V}]. \tag{21}$$

Remark. This does not necessarily imply the restricted field dominance

$$\langle W_C[\mathscr{A}]\rangle_{\mathrm{YM}} = \langle W_C[\mathscr{V}]\rangle_{\mathrm{YM}}, \tag{22}$$

which holds only when the cross term between $\mathscr{V}$ and $\mathscr{X}$ in the action can be neglected.

Finally, we can show that the field strength $F^g_{\mu\nu} := \partial_\mu A^g_\nu - \partial_\nu A^g_\mu$ in NAST is cast into the form:

$$F^g_{\mu\nu}(x) = \sqrt{\frac{2(N-1)}{N}} \mathrm{tr}\{\boldsymbol{n}(x)\mathscr{F}_{\mu\nu}[\mathscr{V}](x)\} + ig^{-1}_{\mathrm{YM}} \mathrm{tr}\{\rho g^\dagger(x)[\partial_\mu, \partial_\nu]g(x)\}. \tag{23}$$

$$\begin{aligned}\mathrm{tr}\{\boldsymbol{n}(x)\mathscr{F}_{\mu\nu}[\mathscr{V}](x)\} =& \partial_\mu \mathrm{tr}\{\boldsymbol{n}(x)\mathscr{V}_\nu(x)\} - \partial_\nu \mathrm{tr}\{\boldsymbol{n}(x)\mathscr{V}_\mu(x)\} \\ &+ \frac{2(N-1)}{N} ig^{-1}_{\mathrm{YM}} \mathrm{tr}\{\boldsymbol{n}(x)[\partial_\mu \boldsymbol{n}(x), \partial_\nu \boldsymbol{n}(x)]\},\end{aligned} \tag{24}$$

where the normalized and traceless **color direction field** $\boldsymbol{n}(x)$ is defined by

$$\boldsymbol{n}(x) = \sqrt{\frac{N}{2(N-1)}} g(x) \left[\rho - \frac{\mathbf{1}}{\mathrm{tr}(\mathbf{1})}\right] g^\dagger(x), \quad g(x) \in G. \tag{25}$$

Thus the Wilson loop operator can be rewritten in terms of new variables:

$$W_C[\mathscr{A}] = \int [d\mu(g)]_\Sigma \exp\Big[- ig_{\mathrm{YM}} \frac{1}{2} \sqrt{\frac{2(N-1)}{N}} \int_{\Sigma:\partial\Sigma=C} 2\mathrm{tr}\{\boldsymbol{n}\mathscr{F}[\mathscr{V}]\}\Big]. \tag{26}$$

Incidentally, the last part $ig^{-1}_{\mathrm{YM}}\mathrm{tr}\{\rho g(x)^\dagger[\partial_\mu, \partial_\nu]g(x)\}$ in $F^g_{\mu\nu}(x)$ corresponds to the **Dirac string**. This term is not gauge invariant and does not contribute to the Wilson loop operator in the end, since it disappears after the group integration $d\mu(g)$ is performed.

§ Magnetic monopoles in Yang-Mills theory

In this way we obtain another expression of the NAST for the Wilson loop operator: For $SU(N)$ in the **fundamental representation**:

$$W_C[\mathscr{A}] = \int [d\mu(g)] \exp \left\{ -ig_{\rm YM} \frac{1}{2} \sqrt{\frac{2(N-1)}{N}} [(N_\Sigma, j) + (\Xi_\Sigma, k)] \right\}, \tag{1}$$

where we have defined the $(D-3)$-form k and one-form j by

$$k := \delta^* f, \quad j := \delta f, \quad f := 2{\rm tr}\{\boldsymbol{n}\mathscr{F}[\mathscr{V}]\}, \tag{2}$$

and we have defined the $(D-3)$-form Ξ_Σ and one-form N_Σ by (Ξ_Σ is the D-dim. solid angle)

$$\Xi_\Sigma := {}^* d\Delta^{-1}\Theta_\Sigma = \delta\Delta^{-1*}\Theta_\Sigma, \quad N_\Sigma := \delta\Delta^{-1}\Theta_\Sigma, \tag{3}$$

with the inner product for the two forms defined by

$$(\Xi_\Sigma, k) = \frac{1}{(D-3)!} \int d^D x k^{\mu_1 \cdots \mu_{D-3}}(x) \Xi_\Sigma^{\mu_1 \cdots \mu_{D-3}}(x), \quad (N_\Sigma, j) = \int d^D x j^\mu(x) N_\Sigma^\mu(x). \tag{4}$$

Thus the Wilson loop operator can be expressed by the electric current j and the monopole current k. The magnetic monopole described by the current k is a topological object of **co-dimension** 3:

- $D = 3$: 0-dimensional point defect $\rightarrow$ point-like magnetic monopole (cf. Wu-Yang type)
- $D = 4$: 1-dimensional line defect $\rightarrow$ magnetic monopole loop (closed loop)

⊙ **SU(2) case**:[Kondo (1998)]

For $SU(2)$, the gauge-invariant magnetic-monopole current $(D-3)$-form k is obtained

$$k = \delta^* f,\ f_{\mu\nu} = 2\mathrm{tr}\{\boldsymbol{n}\mathscr{F}_{\mu\nu}[\mathscr{V}]\} = \partial_\mu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\nu\} - \partial_\nu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\mu\} + ig_{\mathrm{YM}}^{-1} 2\mathrm{tr}\{\boldsymbol{n}[\partial_\mu \boldsymbol{n}, \partial_\nu \boldsymbol{n}]\}. \tag{5}$$

For the fundamental representation of $SU(2)$, the highest-weight state $|\Lambda\rangle$ yields the color field:

$$|\Lambda\rangle = \begin{pmatrix}1\\0\end{pmatrix} \Longrightarrow \rho := |\Lambda\rangle\langle\Lambda| = \begin{pmatrix}1\\0\end{pmatrix}(1,0) = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} \Longrightarrow \rho - \frac{1}{2}\mathbf{1} = \frac{\sigma_3}{2},$$
$$\Longrightarrow \boldsymbol{n}(x) = g(x)\frac{\sigma_3}{2}g(x)^\dagger \in SU(2)/U(1) \simeq S^2 \simeq P^1(\mathbb{C}). \tag{6}$$

The magnetic charge q_m obeys the **quantization condition** a la Dirac:

$$q_m := \int d^3x k^0 = 4\pi g_{\mathrm{YM}}^{-1}\ell,\ \ell \in \mathbb{Z}. \tag{7}$$

This is suggested from a nontrivial Homotopy group of the map $\boldsymbol{n} : S^2 \to SU(2)/U(1)$:

$$\pi_2(SU(2)/U(1)) = \pi_1(U(1)) = \mathbb{Z}. \tag{8}$$

cf. the Abelian magnetic monopole due to 't Hooft-Polyakov associated with the spontaneous breaking $G = SU(2) \to H = U(1)$:

$$\boldsymbol{n}^A \leftrightarrow \hat{\phi}^A(x)/|\hat{\phi}(x)|. \tag{9}$$

⊙ **SU(3) case**:[Kondo (2008)]

The gauge-invariant magnetic-monopole current $(D-3)$-form k is given by

$$k = \delta^* f, \quad f_{\mu\nu} := \partial_\mu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\nu\} - \partial_\nu 2\mathrm{tr}\{\boldsymbol{n}\mathscr{A}_\mu\} + \frac{4}{3} i g_{\mathrm{YM}}^{-1} 2\mathrm{tr}\{\boldsymbol{n}[\partial_\mu \boldsymbol{n}, \partial_\nu \boldsymbol{n}]\}. \tag{10}$$

For the fundamental representation of $SU(3)$, the highest-weight state $|\Lambda\rangle$ yields the color field:

$$|\Lambda\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \rho := |\Lambda\rangle\langle\Lambda| = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Longrightarrow \rho - \frac{1}{3}\mathbf{1} = \frac{-1}{3}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{11}$$

$$\Longrightarrow \boldsymbol{n}(x) = g(x)\frac{-1}{2\sqrt{3}}\begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} g(x)^\dagger \in SU(3)/U(2) \simeq P^2(\mathbb{C}). \tag{12}$$

The matrix $\mathrm{diag.}(-2,1,1)$ is degenerate. Using the Weyl symmetry (global symmetry as a discrete subgroup of color symmetry), it is changed into λ_8. This color field describes a **non-Abelian magnetic monopole**, which corresponds to the spontaneous symmetry breaking $SU(3) \to U(2)$ in the gauge-Higgs model. The magnetic charge obeys the quantization condition:

$$q_m' := \int d^3x k^0 = 2\pi\sqrt{3} g_{\mathrm{YM}}^{-1} n', \ n' \in \mathbb{Z}. \tag{13}$$

This is suggested from a nontrivial Homotopy group of the map $\boldsymbol{n} : S^2 \to SU(3)/U(2)$

$$\pi_2(SU(3)/[SU(2)\times U(1)]) = \pi_1(SU(2)\times U(1)) = \pi_1(U(1)) = \mathbb{Z}. \tag{14}$$

For a **reference state** $|\Lambda\rangle$ of a given representation of a Lie group G, the **maximal stability subgroup** $\tilde{H}$ is defined to be a subgroup leaving $|\Lambda\rangle$ invariant (up to a phase $\phi(h)$):

$$h \in \tilde{H} \Longleftrightarrow h|\Lambda\rangle = |\Lambda\rangle e^{i\phi(h)}. \tag{15}$$

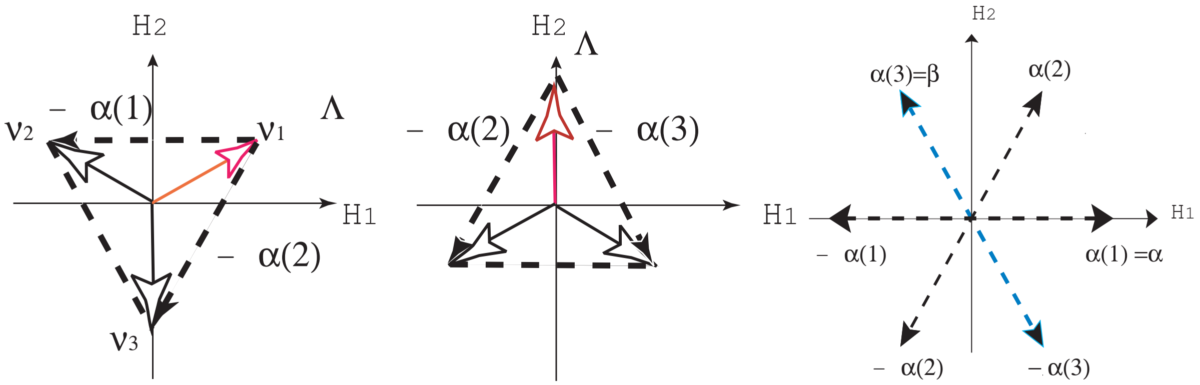
Then a group element g of G is decomposed as

$$g = \xi h \in G, \xi \in G/\tilde{H}, h \in \tilde{H} \Longrightarrow |g, \Lambda\rangle = g|\Lambda\rangle = \xi h|\Lambda\rangle = \xi|\Lambda\rangle e^{i\phi(h)} = |\xi, \Lambda\rangle e^{i\phi(h)}. \tag{16}$$

Every representation R of $SU(3)$ which is specified by the Dynkin index [m,n] belongs to (I) or (II):

(I) [Maximal case] $m \neq 0$ and $n \neq 0 \Longrightarrow \tilde{H} = H = U(1) \times U(1)$. maximal torus
e.g., adjoint rep.[1,1], $\{H_1, H_2\} \in u(1) + u(1)$,

(II) [Minimal case] $m = 0$ or $n = 0 \Longrightarrow \tilde{H} = U(2)$.
This case occurs when the weight vector Λ is orthogonal to some of the root vectors.
e.g., fundamental rep. [1,0], $\{H_1, H_2, E_\beta, E_{-\beta}\} \in u(2)$, where $\Lambda \perp \beta, -\beta$.



§ Field decomposition a la Cho-Duan-Ge-Faddeev-Niemi

We look for the **gauge covariant decomposition**,

$$\mathscr{A}'_\mu(x) = \mathscr{V}'_\mu(x) + \mathscr{X}'_\mu(x). \tag{1}$$

For the condition (ii) to be gauge covariant, the transformation of the color field $\boldsymbol{n}$ given by

$$g(x) \to U(x)g(x) \Longrightarrow \boldsymbol{n}(x) \to \boldsymbol{n}'(x) = U(x)\boldsymbol{n}(x)U^\dagger(x). \tag{2}$$

requires that $\mathscr{X}_\mu(x)$ transforms as an adjoint (matter) field:

$$\mathscr{X}_\mu(x) \to \mathscr{X}'_\mu(x) = U(x)\mathscr{X}_\mu(x)U^\dagger(x). \tag{3}$$

This immediately means that $\mathscr{V}_\mu(x)$ must transform just like the original gauge field $\mathscr{A}_\mu(x)$:

$$\mathscr{V}_\mu(x) \to \mathscr{V}'_\mu(x) = U(x)\mathscr{V}_\mu(x)U^\dagger(x) + ig_{\rm YM}^{-1}U(x)\partial_\mu U^\dagger(x), \tag{4}$$

since $\mathscr{A}_\mu(x) \to \mathscr{A}'_\mu(x) = U(x)\mathscr{A}_\mu(x)U^\dagger(x) + ig_{\rm YM}^{-1}U(x)\partial_\mu U^\dagger(x)$.

These transformation properties impose restrictions on the requirement to be imposed on the restricted field $\mathscr{V}_\mu(x)$. Such a candidate is [covariant constantness of the color field] [which we call the first **defining equation**]:

$$\text{(I)} \quad \mathscr{D}_\mu[\mathscr{V}]\boldsymbol{n} = 0 \quad (\mathscr{D}_\mu[\mathscr{V}] := \partial_\mu - ig_{\rm YM}[\mathscr{V}_\mu, \cdot]), \tag{5}$$

since the covariant derivative transforms in the adjoint way: $\mathscr{D}_\mu[\mathscr{V}(x)] \to U(x)(\mathscr{D}_\mu[\mathscr{V}](x))U^\dagger(x)$.

For $G = SU(2)$, it is shown that the two conditions (I) and (ii),[the **defining equations** for the decomposition] are compatible and determine the decomposition uniquely:

$$
\begin{aligned}
\mathscr{A}_\mu(x) = \mathscr{V}_\mu(x) + \mathscr{X}_\mu(x), \\
\mathscr{V}_\mu(x) =& c_\mu(x)\boldsymbol{n}(x) + ig_{\rm YM}^{-1}[\boldsymbol{n}(x), \partial_\mu \boldsymbol{n}(x)], \quad c_\mu(x) := \mathscr{A}_\mu(x) \cdot \boldsymbol{n}(x), \\
\mathscr{X}_\mu(x) =& -ig_{\rm YM}^{-1}[\boldsymbol{n}(x), \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{n}(x)].
\end{aligned} \tag{6}
$$

This is the same as the **Cho–Duan-Ge (CDG) decomposition** or **Cho–Duan-Ge–Faddeev-Niemi (CDGFN) decomposition** [Cho(1980), Duan-Ge (1979), Faddeev-Niemi (1998)].

The condition (I) means that the field strength $\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x)$ of the field $\mathscr{V}_\mu(x)$ and $\boldsymbol{n}(x)$ commute:

$$
[\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x), \boldsymbol{n}(x)] = 0. \tag{7}
$$

This follows from the identity:

$$
[\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}, \boldsymbol{n}] = ig_{\rm YM}^{-1}[\mathscr{D}_\mu^{[\mathscr{V}]}, \mathscr{D}_\nu^{[\mathscr{V}]}]\boldsymbol{n}, \tag{8}
$$

which is derived from

$$
\mathscr{F}_{\mu\nu}^{[\mathscr{V}]} = ig_{\rm YM}^{-1}[\mathscr{D}_\mu^{[\mathscr{V}]}, \mathscr{D}_\nu^{[\mathscr{V}]}], \quad \mathscr{D}_\mu^{[\mathscr{V}]} := \partial_\mu - ig_{\rm YM}[\mathscr{V}_\mu, \cdot]. \tag{9}
$$

For $SU(2)$, (7) means that $\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x)$ is proportional to $\boldsymbol{n}(x)$:

$$
\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x) = f_{\mu\nu}(x)\boldsymbol{n}(x) \Longrightarrow f_{\mu\nu}(x) = \boldsymbol{n}(x) \cdot \mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x) = 2{\rm tr}[\boldsymbol{n}(x)\mathscr{F}_{\mu\nu}^{[\mathscr{V}]}(x)], \tag{10}
$$

§ Field decomposition for $SU(N)$: new options

For $G = SU(N)$ $(N \geq 3)$, (I) and (ii) are not sufficient to uniquely determine the decomposition. For $G = SU(N)$ $(N \geq 3)$, the condition (ii) must be modified: [Kondo, Shinohara & Murakami(2008)]
(II) $\mathscr{X}^\mu(x)$ does not have the $\tilde{H}$-commutative part, i.e., $\mathscr{X}^\mu(x)_{\tilde{H}} = 0$:

$$\text{(II)} \quad 0 = \mathscr{X}^\mu(x)_{\tilde{H}} := \mathscr{X}^\mu(x) - \frac{2(N-1)}{N}[\boldsymbol{n}(x), [\boldsymbol{n}(x), \mathscr{X}^\mu(x)]]$$

$$\Longleftrightarrow \mathscr{X}^\mu(x) = \frac{2(N-1)}{N}[\boldsymbol{n}(x), [\boldsymbol{n}(x), \mathscr{X}^\mu(x)]]. \tag{1}$$

This condition is also gauge covariant. Note that the condition (ii) follows from (II). For $G = SU(2)$, i.e., $N = 2$, the condition (II) reduces to (ii). By solving (I) and (II), $\mathscr{X}_\mu(x)$ and $\mathscr{V}_\mu(x)$ are determined

$$\mathscr{X}_\mu(x) = -ig_{\rm YM}^{-1}\frac{2(N-1)}{N}[\boldsymbol{n}(x), \mathscr{D}_\mu[\mathscr{A}]\boldsymbol{n}(x)] \in Lie(G/\tilde{H}) = su(N)/u(N-1), \tag{2}$$

$$\mathscr{V}_\mu(x) = \mathscr{C}_\mu(x) + \mathscr{B}_\mu(x) \in \mathscr{L}ie(G) = su(N),$$

$$[\mathscr{C}_\mu(x), \boldsymbol{n}(x)] = 0 \Longleftrightarrow \mathscr{C}_\mu(x) \times \boldsymbol{n}(x) = 0.$$

$$\mathrm{tr}[\mathscr{B}_\mu(x)\boldsymbol{n}(x)] = 0 \Longleftrightarrow \mathscr{B}_\mu(x) \cdot \boldsymbol{n}(x) = 0.$$

$$\mathscr{C}_\mu(x) = \mathscr{A}_\mu(x) - \frac{2(N-1)}{N}[\boldsymbol{n}(x), [\boldsymbol{n}(x), \mathscr{A}_\mu(x)]] \in \mathscr{L}ie(\tilde{H}) = u(N-1),$$

$$\mathscr{B}_\mu(x) = ig_{\rm YM}^{-1}\frac{2(N-1)}{N}[\boldsymbol{n}(x), \partial_\mu\boldsymbol{n}(x)] \in \mathscr{L}ie(G/\tilde{H}) = su(N)/u(N-1), \tag{3}$$

§ Reformulating Yang-Mills theory using new variables

We consider the change of variables from $\mathscr{A}_\mu$ to new field variables $\mathscr{C}_\mu$, $\mathscr{X}_\mu$ and $\boldsymbol{n}$: (See [Kondo, Murakami and Shinohara (2005)] for $SU(2)$, and [Kondo, Shinohara and Murakami (2008)] for $SU(N)$)

$$\mathscr{A}_\mu^A \Longrightarrow (\boldsymbol{n}^\beta, \mathscr{C}_\mu^k, \mathscr{X}_\mu^b), \tag{1}$$

- $\mathscr{A}_\mu \in Lie(G) \to \#[\mathscr{A}_\mu^A] = D \cdot \dim G = D(N^2-1)$
- $\mathscr{C}_\mu \in Lie(\tilde{H}) = u(N-1) \to \#[\mathscr{C}_\mu^k] = D \cdot \dim \tilde{H} = D(N-1)^2$
- $\mathscr{X}_\mu \in Lie(G/\tilde{H}) \to \#[\mathscr{X}_\mu^b] = D \cdot \dim(G/\tilde{H}) = D(2N-2)$
- $\boldsymbol{n} \in Lie(G/\tilde{H}) \to \#[\boldsymbol{n}^\beta] = \dim(G/\tilde{H}) = 2(N-1)$.

The new theory written in terms of new variables $(\boldsymbol{n}^\beta, \mathscr{C}_\mu^k, \mathscr{X}_\mu^b)$ has the $2(N-1)$ extra degrees of freedom. Therefore, we must give a procedure for eliminating the $2(N-1)$ extra degrees of freedom to obtain the new theory which is equipollent to the original one.
For this purpose, we impose $2(N-1)$ constraints $\boldsymbol{\chi} = 0$, which we call the **reduction condition**:

- $\boldsymbol{\chi} \in Lie(G/\tilde{H}) \to \#[\boldsymbol{\chi}^a] = \dim(G/\tilde{H}) = 2(N-1) = \#[\boldsymbol{n}^\beta]$.

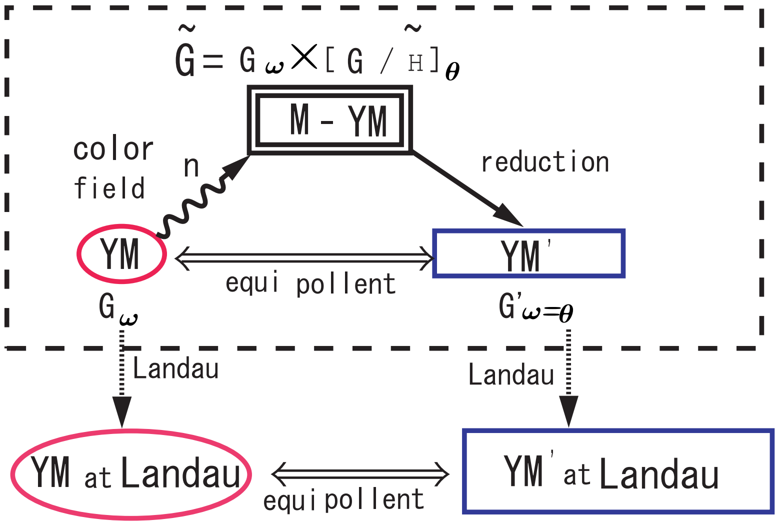


Figure 2: The relationship between the original Yang-Mills (YM) theory and the reformulated Yang-Mills (YM') theory. A single color field $\boldsymbol{n}$ is introduced to enlarge the original Yang-Mills theory with a gauge group G into the master Yang-Mills (M-YM) theory with the enlarged gauge symmetry $\tilde{G} = G \times G/\tilde{H}$. The reduction conditions are imposed to reduce the master Yang-Mills theory to the reformulated Yang-Mills theory with the equipollent gauge symmetry G'. In addition, we can impose any over-all gauge fixing condition, e.g., Landau gauge to both the original YM theory and the reformulated YM' theory.

- Enlarged gauge symmetry by introducing n and the reduction by imposing χ

$$G \xrightarrow{\boldsymbol{n}} G \times G/\tilde{H} \xrightarrow{\boldsymbol{\chi}} G. \tag{2}$$

A choice of the reduction condition in the minimal option is to minimize the functional $F_{\rm red}[\mathscr{A}, \boldsymbol{n}]$:

$$F_{\rm red}[\mathscr{A}, \boldsymbol{n}] = \int d^D x \frac{1}{2} g^2 \mathscr{X}_\mu \cdot \mathscr{X}^\mu = \frac{N-1}{N} \int d^D x (D_\mu[\mathscr{A}]\boldsymbol{n})^2,$$

with respect to the enlarged gauge transformation:

$$\begin{aligned} \delta \mathscr{A}_\mu &= D_\mu[\mathscr{A}]\boldsymbol{\omega} \quad (\boldsymbol{\omega} \in \mathscr{L}ie(G)), \\ \delta \boldsymbol{n} &= ig[\boldsymbol{n}, \boldsymbol{\theta}] = ig[\boldsymbol{n}, \boldsymbol{\theta}_\perp] \quad (\boldsymbol{\theta}_\perp \in \mathscr{L}ie(G/\tilde{H})). \end{aligned} \tag{3}$$

In fact, the enlarged gauge transformation of the functional $F_{\rm red}[\mathscr{A}, \boldsymbol{n}]$ is

$$\delta F_{\rm red}[\mathscr{A}, \boldsymbol{n}] = \delta \int d^D x \frac{1}{2} (D_\mu[\mathscr{A}]\boldsymbol{n})^2 = g \int d^D x (\boldsymbol{\theta}_\perp - \boldsymbol{\omega}_\perp) \cdot i[\boldsymbol{n}, D^\mu[\mathscr{A}]D_\mu[\mathscr{A}]\boldsymbol{n}], \tag{4}$$

where $\boldsymbol{\omega}_\perp$ denotes the component of $\boldsymbol{\omega}$ in the direction $\mathscr{L}(G/\tilde{H})$.
For $\boldsymbol{\omega}_\perp = \boldsymbol{\theta}_\perp$ (diagonal part of $G \times G/\tilde{H}$) $\delta F_{\rm red}[\mathscr{A}, \boldsymbol{n}] = 0$ imposes no condition, while for $\boldsymbol{\omega}_\perp \neq \boldsymbol{\theta}_\perp$ (off-diagonal part of $G \times G/\tilde{H}$) it implies the constraint:

$$\boldsymbol{\chi}[\mathscr{A}, \boldsymbol{n}] := [\boldsymbol{n}, D^\mu[\mathscr{A}]D_\mu[\mathscr{A}]\boldsymbol{n}] \equiv 0. \tag{5}$$

Note that the number of constraint is $\#[\boldsymbol{\chi}] = \dim(G \times G/\tilde{H}) - \dim(G) = \dim(G/\tilde{H})$ as desired. Finally, we have an equipollent Yang-Mills theory with **the residual local gauge symmetry** $G' := SU(N)^{\rm local}_{\omega'}$ with the gauge transformation parameter:

$$\boldsymbol{\omega}'(x) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\omega}_\perp(x)) = (\boldsymbol{\omega}_\parallel(x), \boldsymbol{\theta}_\perp(x)), \quad \boldsymbol{\omega}_\perp(x) = \boldsymbol{\theta}_\perp(x). \tag{6}$$

	original YM	$\Longrightarrow$ reformulated YM
field variables	$\mathscr{A}_\mu^A \in \mathscr{L}(G)$	$\Longrightarrow \boldsymbol{n}^\beta, \mathscr{C}_\nu^k, \mathscr{X}_\nu^b$
action	$S_{\rm YM}[\mathscr{A}]$	$\Longrightarrow \tilde{S}_{\rm YM}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}]$
integration measure	$\mathcal{D}\mathscr{A}_\mu^A$	$\Longrightarrow \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J} \delta(\tilde{\boldsymbol{\chi}}) \Delta_{\rm FP}^{\rm red}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}]$

At the same time, the color field

$$\boldsymbol{n}(x) \in \mathscr{L}ie(G/\tilde{H})$$

must be obtained by solving the reduction condition $\boldsymbol{\chi} = 0$ for a given $\mathscr{A}$, e.g.,

$$\boldsymbol{\chi}[\mathscr{A}, \boldsymbol{n}] := [\boldsymbol{n}, D^\mu[\mathscr{A}] D_\mu[\mathscr{A}] \boldsymbol{n}] \in \mathscr{L}ie(G/\tilde{H}). \tag{7}$$

Here $\tilde{\boldsymbol{\chi}} = 0$ is the reduction condition written in terms of the new variables:

$$\tilde{\boldsymbol{\chi}} := \tilde{\boldsymbol{\chi}}[\boldsymbol{n}, \mathscr{C}, \mathscr{X}] := D^\mu[\mathscr{V}] \mathscr{X}_\mu, \tag{8}$$

and $\Delta_{\rm FP}^{\rm red}$ is the Faddeev-Popov determinant associated with the reduction condition:

$$\Delta_{\rm FP}^{\rm red} := \det \left(\frac{\delta \boldsymbol{\chi}}{\delta \boldsymbol{\theta}} \right)_{\boldsymbol{\chi}=0} = \det \left(\frac{\delta \boldsymbol{\chi}}{\delta \boldsymbol{n}^\theta} \right)_{\boldsymbol{\chi}=0}. \tag{9}$$

which is obtained by the BRST method as $\Delta_{\rm FP}^{\rm red}[\boldsymbol{n}, c, \mathscr{X}] = \det\{-D_\mu[\mathscr{V} + \mathscr{X}] D_\mu[\mathscr{V} - \mathscr{X}]\}$.
The Jacobian $\tilde{J}$ is very simple, irrespective of the choice of the reduction condition:

$$\tilde{J} = 1. \tag{10}$$

[Kondo, Shinohara & Murakami, Prog.Theor.Phys. **120**, 1–50 (2008). arXiv:0803.0176]

The Wilson loop average in the original theory:

$$W(C) := \langle W_C[\mathscr{A}] \rangle_{\rm YM} = Z_{\rm YM}^{-1} \int \mathcal{D}\mathscr{A} e^{-S_{\rm YM}[\mathscr{A}]} W_C[\mathscr{A}]. \tag{11}$$

is defined in the reformulated Yang-Mills theory:

$$\begin{aligned}\langle W_C[\mathscr{A}] \rangle_{\rm YM'} =& Z_{\rm YM'}^{-1} \int [d\mu(g)] \int \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J} \delta(\tilde{\boldsymbol{\chi}}) \Delta_{\rm FP}^{\rm red} e^{-\tilde{S}_{\rm YM}[\boldsymbol{n},\mathscr{C},\mathscr{X}]} \\ & \times \exp\left\{ i g_{\rm YM} \sqrt{\frac{2(N-1)}{N}} [(j, N_\Sigma) + (k, \Xi_\Sigma)] \right\}, \\ Z_{\rm YM'} =& \int \mathcal{D}n^\beta \mathcal{D}\mathscr{C}_\nu^k \mathcal{D}\mathscr{X}_\nu^b \tilde{J} \delta(\tilde{\boldsymbol{\chi}}) \Delta_{\rm FP}^{\rm red} e^{-S_{\rm YM'}[\boldsymbol{n},\mathscr{C},\mathscr{X}]}.\end{aligned} \tag{12}$$

Remark:

1. For $SU(2)$, when we fix the color field $\mathbf{n}(x) = (0,0,1)$ or $\boldsymbol{n}(x) = \sigma_3/2$, the reduction condition $D^\mu[\mathscr{V}]\mathscr{X}_\mu = 0$ reduces to the conventional **Maximally Abelian gauge (MA gauge)**.
2. For $SU(3)$, this is not the case: This reduction does not reduce to the conventional Maximally Abelian gauge for $SU(3)$, even if the color field is fixed to be uniform. Therefore, the results to be obtained are nontrivial.
3. For $\Omega_\mu(x) := i g_{\rm YM}^{-1} g(x) \partial_\mu g^\dagger(x)$, $\partial_\mu \boldsymbol{n}(x) = i g_{\rm YM} [\Omega_\mu(x), \boldsymbol{n}(x)]$, i.e., $\mathscr{D}[\Omega]\boldsymbol{n}(x) = 0$.
It is shown $\Omega_\mu(x) = \mathscr{B}_\mu(x) + a_\mu(x)$ where $[a_\mu(x), \boldsymbol{n}(x)] = 0$. Therefore, $\mathscr{D}[\mathscr{B}]\boldsymbol{n}(x) = 0$. For $\mathscr{V}_\mu(x) = \mathscr{B}_\mu(x) + \mathscr{C}_\mu(x)$, thus, $\mathscr{D}[\mathscr{V}]\boldsymbol{n}(x) = 0$, since $[\mathscr{C}_\mu(x), \boldsymbol{n}(x)] = 0$.

Part II: Lattice reformulation and Numerical results for SU(3) case

For the summary of this part, see
A. Shibata, K.-I. Kondo, S. Kato and T. Shinohara, arXiv:1412.8009 [hep-lat]

Lattice reformulation and Numerical simulations SU(2)

- Ito, Kato, Kondo, Murakami, Shibata and Shinohara, Phys.Lett.B 645, 67(2007).
- Shibata, Kato, Kondo, Murakami, Shinohara and Ito, Phys. Lett. B653, 101 (2007).

Lattice reformulation SU(N)

- Kondo, Shibata, Shinohara, Murakami, Kato and Ito, Phys. Lett. B**669**, 107(2008)
- Shibata, Kondo and Shinohara, Phys. Lett. B**691**, 91(2010)

Numerical simulations SU(3)

- Kondo, Shibata, Shinohara and Kato, Phys. Rev. D**83**, 114016 (2011)
- Shibata, Kondo, Shinohara and Kato, Phys. Rev. D**87**, 054011 (2013).

§ Numerical simulations for $SU(3)$: quark potential

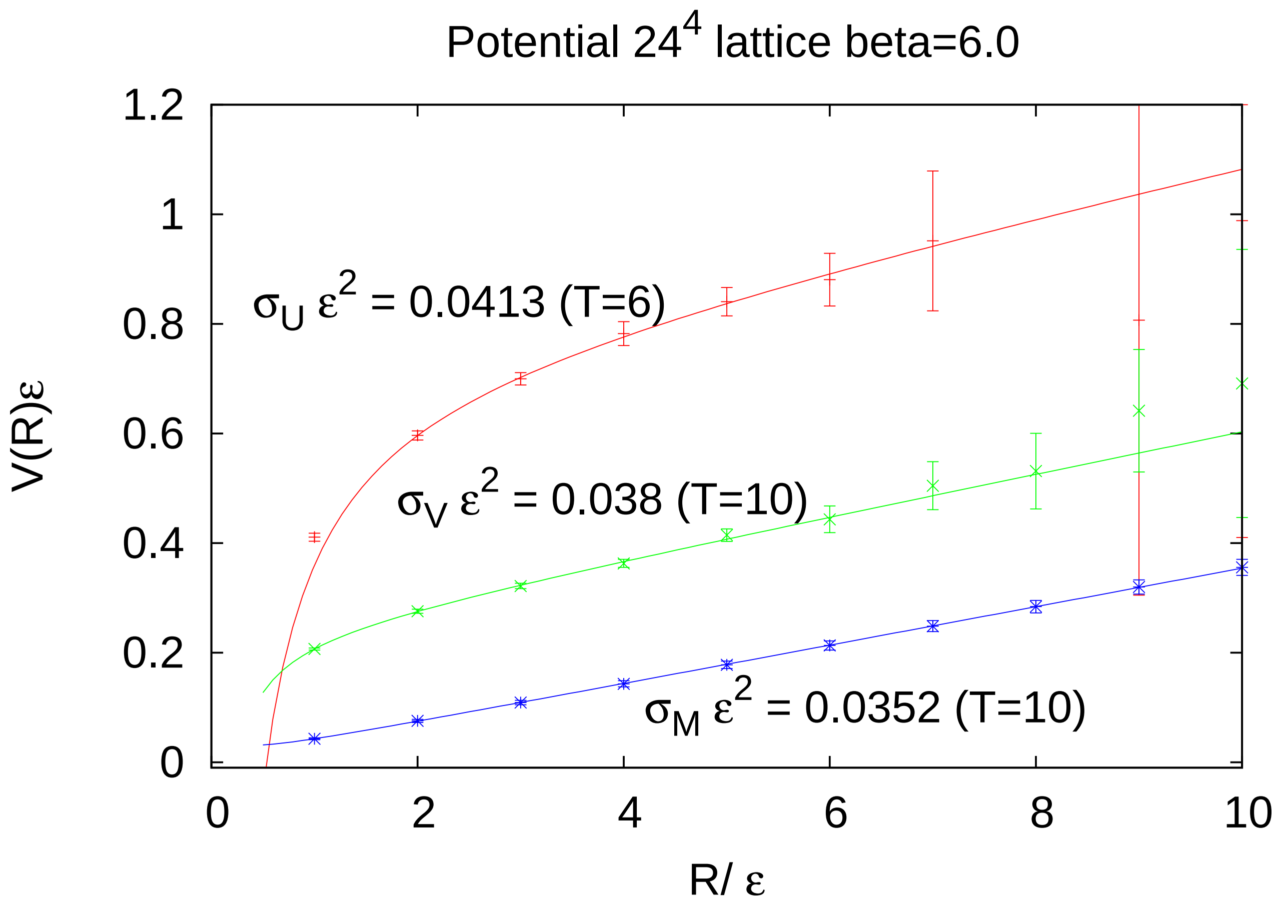


Figure 3: $SU(3)$ quark-antiquark potentials as functions of the quark-antiquark distance R: (from above to below) (i) full potential $V_{\rm full}(R)$ (red curve), (ii) restricted part $V_{\rm rest}(R)$ (green curve) and (iii) magnetic–monopole part $V_{\rm mono}(R)$ (blue curve), measured at $\beta = 6.0$ on 24^4 using 500 configurations where ϵ is the lattice spacing.

The results of our numerical simulations exhibit the **infrared restricted variable $\mathscr{V}$ dominance** in the string tension, e.g.,

$$\frac{\sigma_{\rm rest}}{\sigma_{\rm full}} = \frac{0.0380}{0.0413} \simeq 0.92,$$

and the **non-Abelian magnetic monopole dominance** in the string tension, e.g.,

$$\frac{\sigma_{\rm mono}}{\sigma_{\rm full}} = \frac{0.0352}{0.0413} \simeq 0.85.$$

However, we know that $\sigma_{\rm full}$ has the largest errors among three string tensions. Incidentally, if we use the other data for $\epsilon\sqrt{\sigma_{\rm full}*}$ at $\beta = 6.0$ where $\epsilon^2\sigma_{\rm full}* = (\epsilon\sqrt{\sigma_{\rm full}*})^2 = 0.2154^2 \sim 0.2209^2 = 0.0464 \sim 0.0488$, the ratios of two string tensions $\sigma_{\rm rest}, \sigma_{\rm mono}$ to the total string tension $\sigma_{\rm full}$ are modified.

Thus, we have obtained the **infrared restricted variable $\mathscr{V}$ dominance** in the string tension and the **non-Abelian magnetic monopole dominance** in the string tension. Both dominance are obtained in the gauge independent way.

§ Chromoelectric field and flux tube formation

In order to extract the chromo-field created by a quark-antiquark pair, we use a gauge-invariant a gauge-invariant connected correlator between a plaquette and the Wilson loop proposed by Di Giacomo, Maggiore and Olejnik (1990): (see Fig.4):

$$\rho_{U_P} := \frac{\left\langle \mathrm{tr}\left(U_P L^\dagger W L\right)\right\rangle}{\langle \mathrm{tr}\,(W)\rangle} - \frac{1}{N}\frac{\langle \mathrm{tr}\,(U_P)\,\mathrm{tr}\,(W)\rangle}{\langle \mathrm{tr}\,(W)\rangle},$$

In the continuum limit, ρ_{U_P} reduces to the field strength in presence of $q\bar{q}$ source:

$$\rho_{U_P} \overset{\varepsilon\to 0}{\simeq} g\epsilon^2 \left\langle \mathscr{F}_{\mu\nu}\right\rangle_{q\bar{q}} := \frac{\left\langle \mathrm{tr}\left(ig\epsilon^2 \mathscr{F}_{\mu\nu} L^\dagger W L\right)\right\rangle}{\langle \mathrm{tr}\,(W)\rangle} + O(\epsilon^4),$$

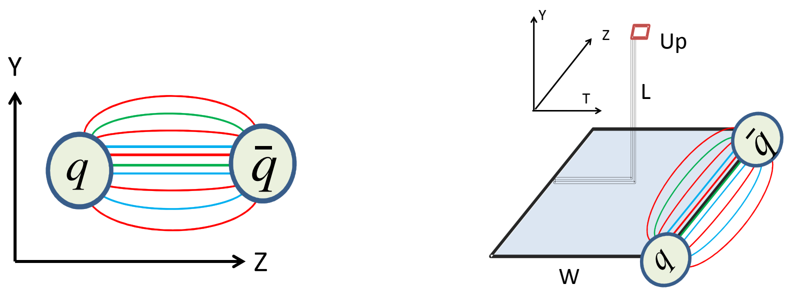


Figure 4: (Left) The setup of measuring the chromo-flux produced by a quark–antiquark pair. (Right) The gauge-invariant connected correlator $(U_p L W L^\dagger)$ between a plaquette U and the Wilson loop W.

Thus, the **gauge-invariant chromo-field strength** $F_{\mu\nu}[U]$ produced by a $q\bar{q}$ pair is given by $F_{\mu\nu}[U] := \epsilon^{-2}\sqrt{\frac{\beta}{2N}}\rho_{U_P}$.

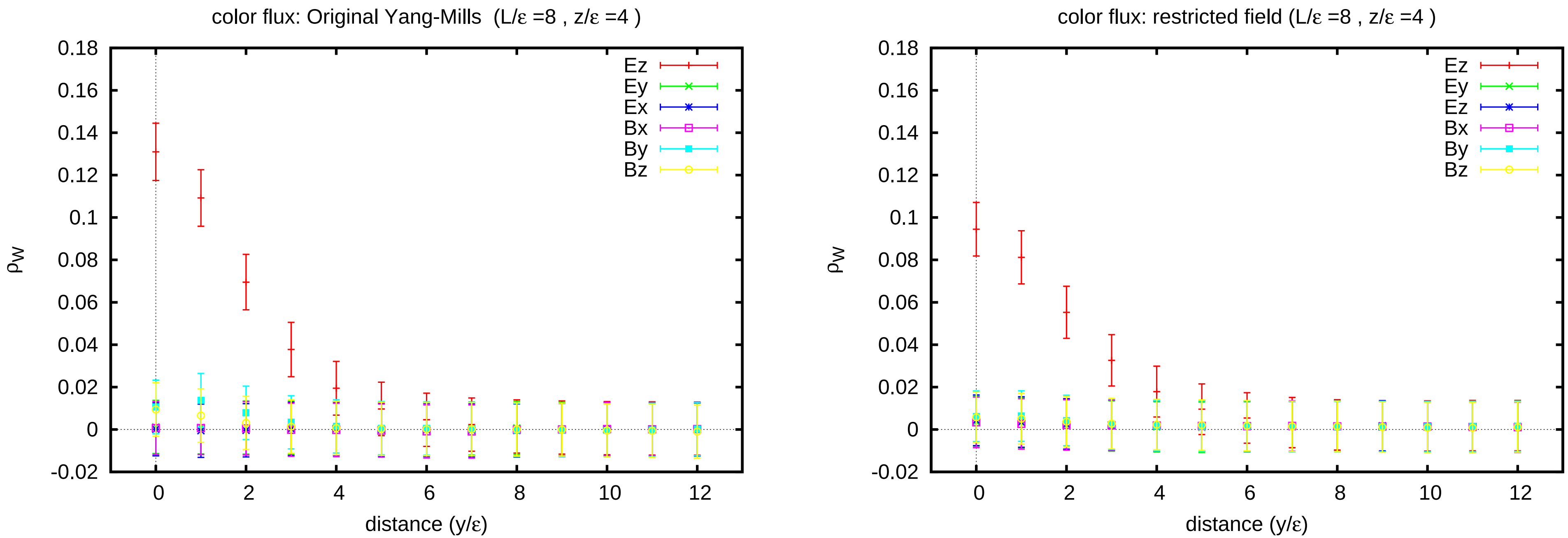


Figure 5: Measurement of components of the chromoelectric field $\boldsymbol{E}$ and chromomagnetic field $\boldsymbol{B}$ as functions of the distance y from the z axis. (Left panel) the original $SU(3)$ Yang-Mills field, (Right panel) the restricted field.

From Fig.5 we find that only the E_z component of the **chromoelectric field** $(E_x, E_y, E_z) = (F_{10}, F_{20}, F_{30})$ connecting q and $\bar{q}$ has non-zero value for both the restricted field V and the original Yang-Mills field U.
The other components are zero consistently within the numerical errors. This means that the chromomagnetic field $(B_x, B_y, B_z) = (F_{23}, F_{31}, F_{12})$ connecting q and $\bar{q}$ does not exist and that the chromoelectric field is parallel to the z axis on which quark and antiquark are located.

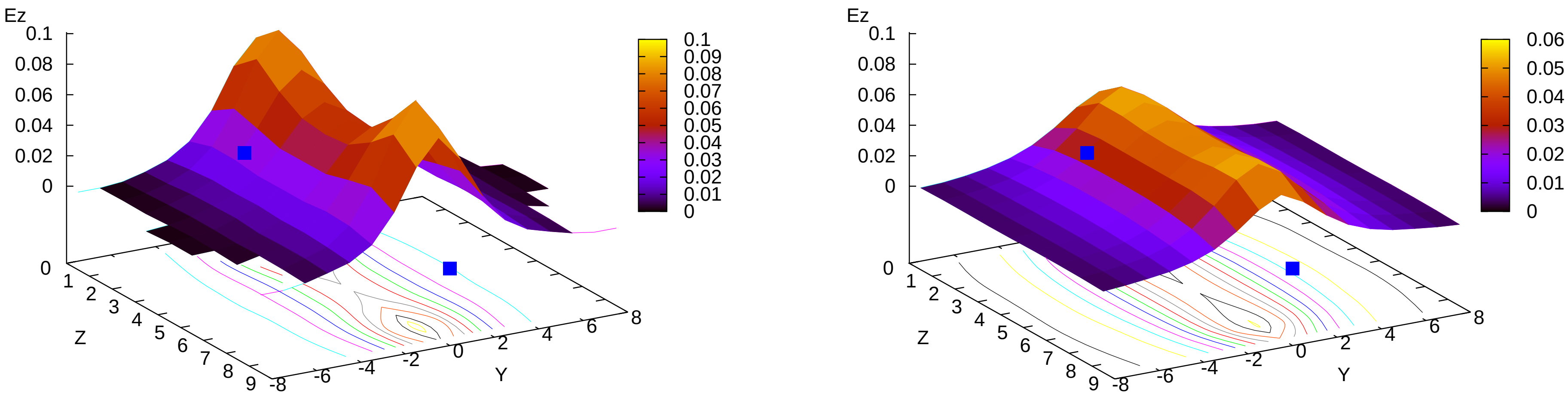


Figure 6: The distribution in Y-Z plane of the chromoelectric field E_z connecting a pair of quark and antiquark: (Left panel) chromoelectric field produced from the original Yang-Mills field, (Right panel) chromoelectric field produced from the restricted field.

§ Magnetic current and dual Meissner effect

The dual Meissner effect is examined by the simultaneous formation of the chromoelectric flux tube and the associated magnetic-monopole current induced around it. The magnetic-monopole current $\mathbf{k}$ induced around the flux can be calculated as $k = \delta^* F[V] = {}^*dF[V]$,

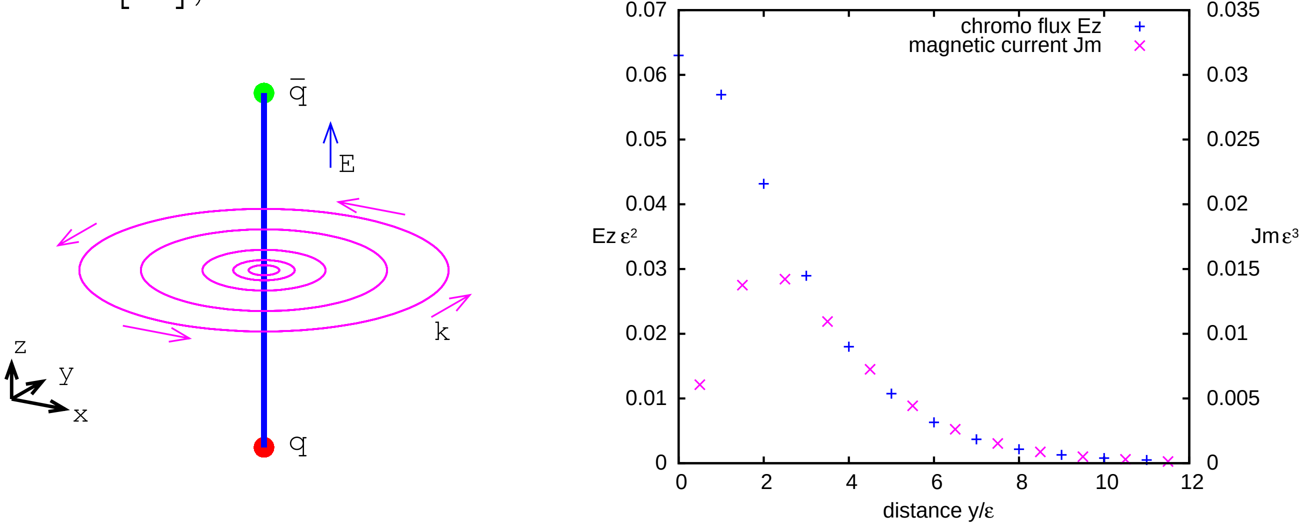


Figure 7: The magnetic-monopole current $\mathbf{k}$ induced around the flux along the z axis connecting a quark-antiquark pair. (Left panel) The positional relationship between the chromoelectric field E_z and the magnetic current $\mathbf{k}$. (Right panel) The magnitude of the chromo-electronic current E_z and the magnetic current $J_m = |\mathbf{k}|$ as functions of the distance y from the z axis.

These numerical evidences support "non-Abelian" dual superconductivity due to non-Abelian magnetic monopoles as a mechanism for confinement in $SU(3)$ Yang-Mills theory.

§ Type of dual superconductivity

The fitting of our data to the **Ginzburg-Landau (GL) theory** shows that the dual superconductor of $SU(3)$ Yang-Mills theory is type I:

$$\kappa = 0.45 \pm 0.01 < \kappa_c = 1/\sqrt{2} \simeq 0.707.$$

The penetration length $\lambda = 0.1207(17)$fm and the coherence length $\xi = 0.2707(86)$fm is obtained in units of the string tension $\sigma_{\text{phys}} = (440\text{MeV})^2$,
The restricted field dominance: the restricted part plays the dominant role $\kappa = 0.48 \pm 0.02$,$\lambda = 0.132(3)$fm and $\xi = 0.277(14)$fm.

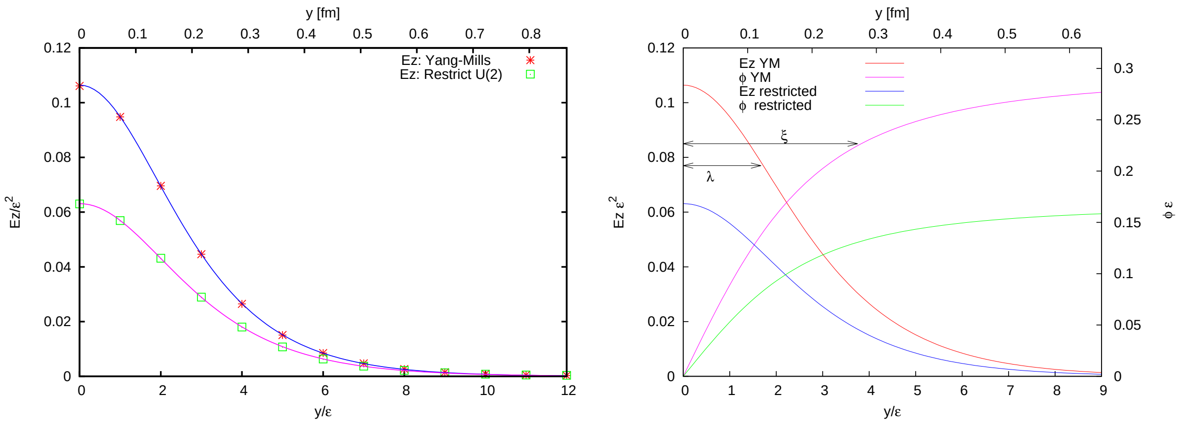


Figure 8: (Left panel) The plot of the chromoelectric field E_z versus the distance y in units of the lattice spacing ϵ and the fitting as a function $E_z(y)$ of y according to (23). The red cross for the original $SU(3)$ field and the green square symbol for the restricted field. (Right panel) The order parameter ϕ reproduced as a function $\phi(y)$ of y according to (23), together with the chromoelectric field $E_z(y)$.

§ Color direction field and color symmetry

The correlators $\langle n^A(x) n^B(0) \rangle$ are of the form:

$$\langle n^A(x) n^B(0) \rangle = \delta^{AB} D(r) \quad (A, B = 1, 2, \cdots, 8).$$

We have also checked that one-point functions vanish, $\langle n^A(x) \rangle = \pm 0.002 \simeq 0$.

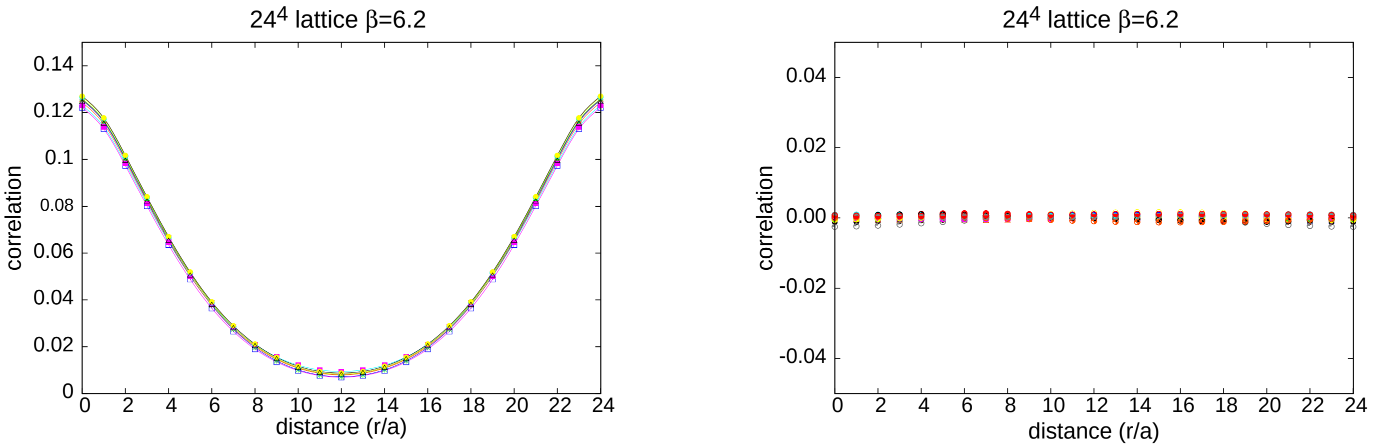


Figure 9: Color field correlators $\langle n^A(x) n^B(0) \rangle$ $(A, B = 1, \cdots, 8)$ as functions of the distance $r := |x|$ measured at $\beta = 6.2$ on 24^4 lattice, using 500 configurations under the Landau gauge. (Left panel) $A = B$, (Right panel) $A \neq B$.

Fig.38 shows the two-point correlation functions of the color field, indicating the global $SU(3)$ **color symmetry** is preserved, that is to say, there is no specific direction in color space.

§ Gluon propagator, infrared dominance and massive (high-energy) mode

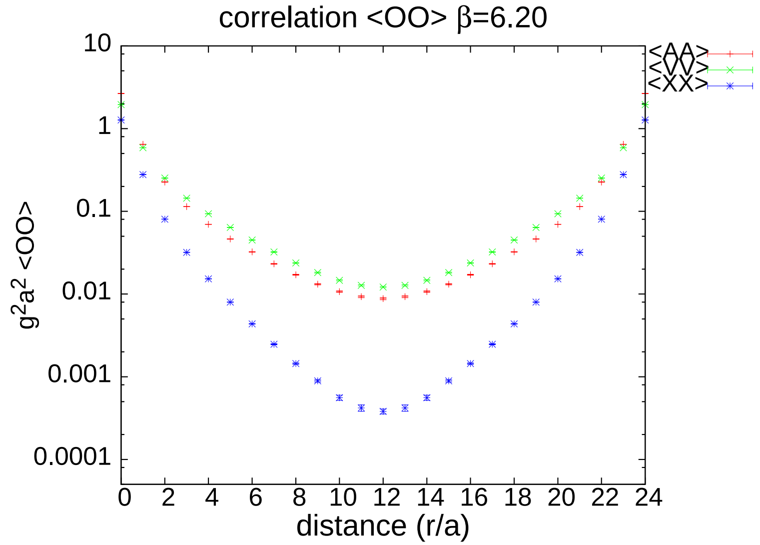


Figure 10: Field correlators as functions of the distance $r := |x|$ (from above to below) $\langle \mathscr{V}_\mu^A(0)\mathscr{V}_\mu^A(r)\rangle$, $\langle \mathscr{A}_\mu^A(0)\mathscr{A}_\mu^A(r)\rangle$, and $\langle \mathscr{X}_\mu^A(0)\mathscr{X}_\mu^A(r)\rangle$.

Fig. 10 shows correlators of the new fields $\mathscr{V}$, $\mathscr{X}$, and the original fields $\mathscr{A}$, indicating the infrared dominance of restricted correlation functions $\langle \mathscr{V}_\mu^A(0)\mathscr{V}_\mu^A(r)\rangle$ in the sense that the variable $\mathscr{V}$ is dominant in the long distance, while the correlator $\langle \mathscr{X}_\mu^A(0)\mathscr{X}_\mu^A(r)\rangle$ of $SU(3)/U(2)$ variable $\mathscr{X}$ decreases quickly.

For $\mathscr{X}$, at least, we can introduce a gauge-invariant "mass" term:

$$\frac{1}{2}M_X^2\mathscr{X}_\mu^A\mathscr{X}_\mu^A,$$

since $\mathscr{X}$ transforms like an adjoint matter field under the gauge transformation. The naively estimated "mass" of $\mathscr{X}$ is $M_X = 2.409\sqrt{\sigma_{\rm phys}} = 1.1$ GeV. This value should be compared with the result in MA gauge.

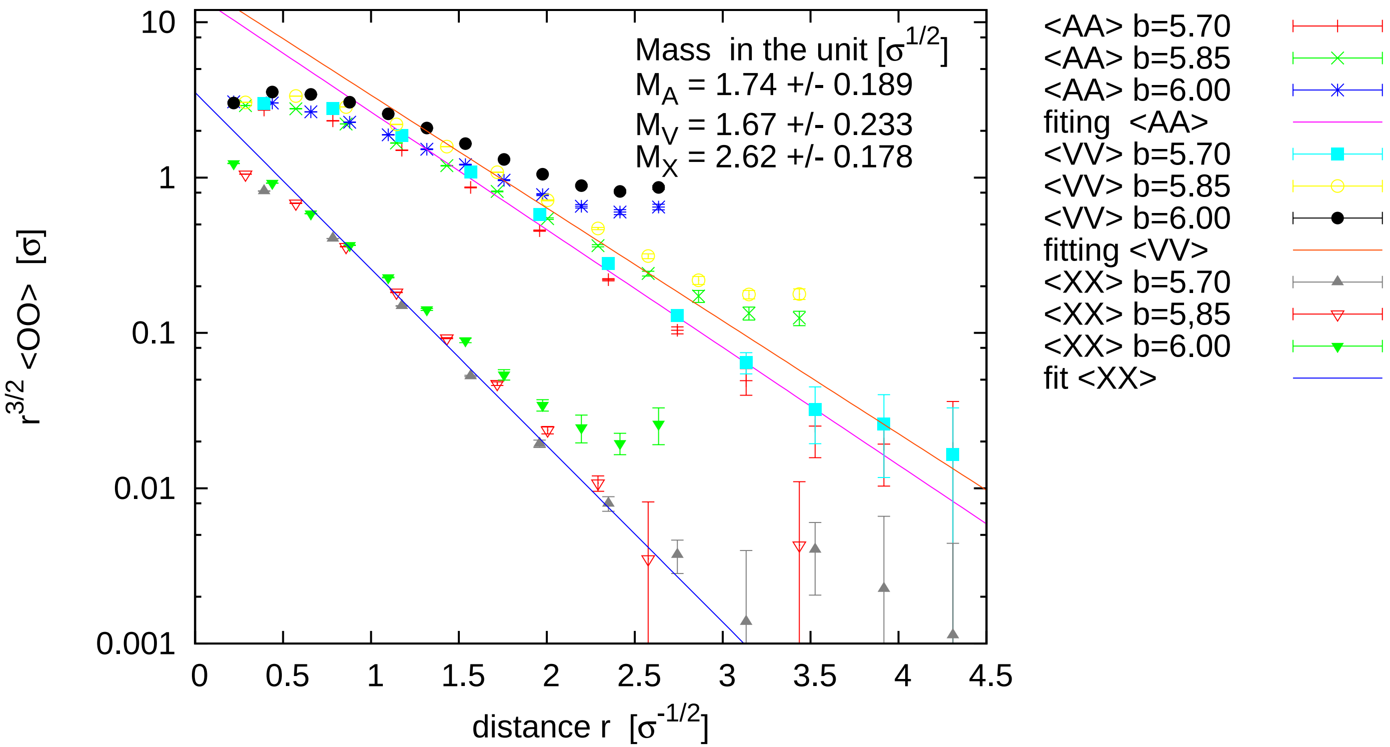


Figure 11: The rescaled correlation correlation functions $r^{3/2}\langle O(r)O(0)\rangle$ for $O = \mathbb{A}, \mathbb{V}, \mathbb{X}$ for 24^4 lattice with $\beta = 5.7,\ 5.85,\ 6.0$. The physical scale is set in units of the string tension $\sigma_{\rm phys}^{1/2}$. The correlation functions have the profile of cosh type because of the periodic boundary condition, and hence we use data within distance of the half size of lattice.

§ magnetic-monopole loops for SU(3)

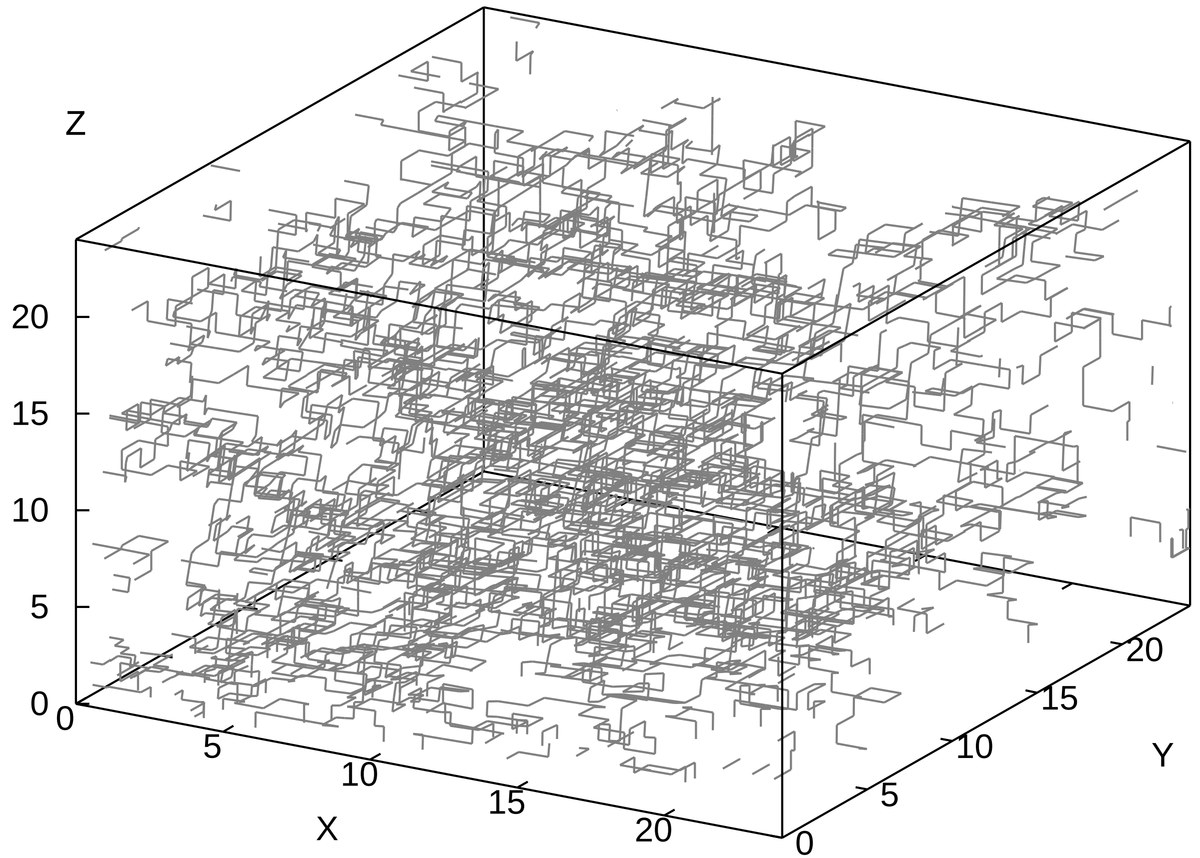


Figure 12: The magnetic-monopole loops on the 4 dimensional lattice where the 3-dimensional plot is obtained by projecting the 4-dimensional dual lattice space to the 3-dimensional one, i.e., $(x, y, z, t) \to (x, y, z)$.

§ Instantons to magnetic monopole via reduction for SU(2)

By solving the reduction condition, a circular loop of monopole current k has been obtained from the Jackiw-Nohl-Rebbi two-instanton configuration:

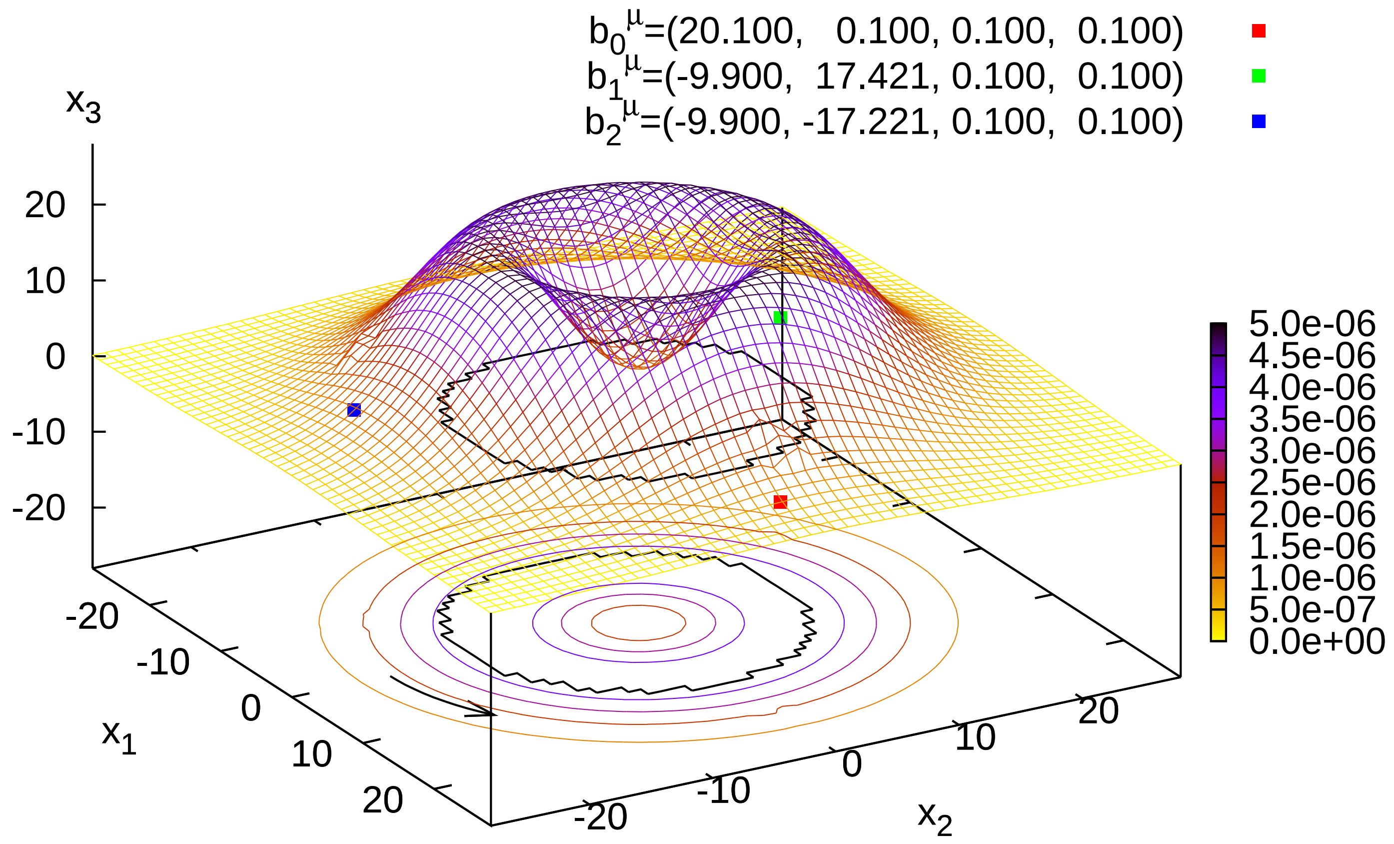


Figure 13: JNR two-instanton and the associated circular magnetic current $k_{x,\mu}$.

Fukui, Kondo, Shibata and Shinohara, Phys.Rev.D82,045015 (2010). arXiv:1005.3157 [hep-th], Jackiw-Nohl-Rebbi two-instanton as a source of magnetic monopole loop.

§ Merons to magnetic monopole via reduction for SU(2)

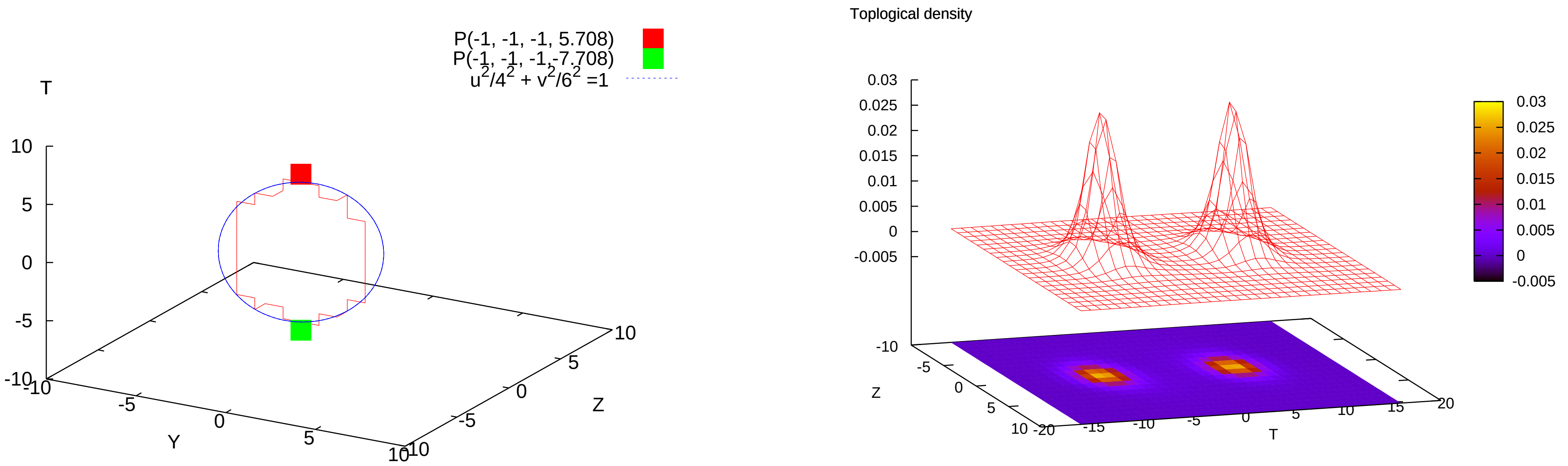


Figure 14: (Left panel) The plot of a magnetic-monopole loop generated by a pair of (smeared) merons in 4-dimensional Euclidean space. The 3-dimensional plot is obtained by projecting the 4-dimensional dual lattice space to the 3-dimensional one, i.e., $(x, y, z, t) \to (y, z, t)$. The positions of two meron sources are described by solid boxes, and the monopole loop by red solid line. A circle of blue line is written for guiding eyes. (Right panel) The plot of the topological charge density for $z-t$ plane (slice of $x = y = 0$). Two peaks of the topological charge density are located at the positions of two merons.

K.-I. Kondo, N. Fukui, A. Shibata and T. Shinohara, Phys. Rev. D**78**, 065033 (2008). arXiv:0806.3913 [hep-th], A. Shibata, K.-I. Kondo, S. Kato, S. Ito, T. Shinohara, and N. Fukui, PoS LAT2009:232,2009. arXiv:0911.4533 [hep-lat]

§ A summary

In order to study quark confinement from the viewpoint of the dual superconductivity, we have combined a non-Abelian Stokes theorem for the Wilson loop operator and the reformulations of the Yang-Mills theory using the new variables.

1) Even in the $SU(N)$ Yang-Mills theory (without adjoint scalar fields), we do not need to use the prescription called the **Abelian projection** ['t Hooft,1981] which realizes magnetic monopoles as **gauge-fixing defects**.

We can define a gauge-invariant (chromo)magnetic monopole k inherent in the non-Abelian Wilson loop operator by using a **non-Abelian Stokes theorem**.
For the $G = SU(2)$ gauge group, the resulting magnetic monopole coincide with one obtained from the CDG decomposition for the Yang-Mills field which was proposed by [Cho (1980)] and [Duan & Ge (1979)] independently. For the $G = SU(2)$ gauge group, such an Abelian magnetic monopole is described by the color field $\boldsymbol{n}(x)$ with the target space: $\boldsymbol{n}(x) \in SU(2)/U(1) = P^1(\mathbb{C})$ for quarks in any representation. However, $G = SU(2)$ is an exceptional case. For $SU(N)$ $(N \geq 3)$, the resulting magnetic monopole depends on the representation of quarks defining the Wilson loop operator.

For the $G = SU(3)$ gauge group, every representation of $SU(3)$ is specified by the Dynkin index $[m, n]$ and the magnetic monopoles are exhausted by two cases:

- For quarks in the representation $m = 0$ or $n = 0$, $\tilde{H} = U(2)$, e.g., the fundamental representation of $G = SU(3)$,

  a non-Abelian magnetic monopole described by $\boldsymbol{n}(x) \in SU(3)/U(2) = P^2(\mathbb{C})$

- For quarks in the representation $m \neq 0$ and $n \neq 0$, $\tilde{H} = H = U(1) \times U(1)$, e.g., the adjoint representation of $G = SU(3)$,

  two Abelian magnetic monopoles described by $\boldsymbol{n}(x) \in SU(3)/[U(1) \times U(1)] = F_2$

Here $\tilde{H}$ is a subgroup of G called the maximal stability group which is uniquely determined once the representation is specified. $\tilde{H}$ does not necessarily agree with the maximal torus group $H = U(1)^{N-1}$.

2) We have constructed a new reformulation of Yang-Mills theory using new field variables obtained by change of variables, which gives an optimal description of the magnetic monopole derived in 1). The reformulation allows a number of options discriminated by the maximal stability group $\tilde{H}$ of the gauge group G. Our reformulations introduce only a single color field $\boldsymbol{n}(x)$ for any N.

The idea of using new variables is originally due to [Cho (1980)] and [Faddeev & Niemi (1999)], where $N-1$ color fields $\boldsymbol{n}_{(j)}$ $(j = 1, ..., N-1)$ are introduced.

However, our reformulation in the minimal option is new for $SU(N), N \geq 3$: we introduce only a single color field $\boldsymbol{n}$ for any N, which is enough for reformulating the quantum Yang-Mills theory to describe confinement of the fundamental quark. The reformulation allows options discriminated by the maximal stability group $\tilde{H}$.

For $G = SU(3)$, two options are possible:

• The maximal option with $\tilde{H} = H = U(1) \times U(1)$, the reformulation gives a manifestly gauge-independent extension of the conventional Abelian projection in the maximal Abelian gauge. This is just the case of Cho and Faddeev & Niemi.

• The minimal option with $\tilde{H} = U(2)$ gives an optimized description of quark confinement through the Wilson loop operator in the fundamental representation. [Kondo, Shinohara and Murakami, 2008] The minimal option in our reformulation is new for $SU(N), N \geq 3$:

We have constructed the lattice versions of the reformulations of the $SU(N)$ Yang-Mills theory and performed numerical simulations on a lattice for $SU(2)$ and $SU(3)$.
3) We have confirmed the infrared dominance of the restricted variables $\mathscr{V}$ and the magnetic monopole dominance for confinement of quarks in the fundamental representation (in the string tension).
Abelian magnetic monopole for SU(2), and non-Abelian magnetic monopole for SU(3).
cf. [infrared Abelian dominance and magnetic monopole dominance in MA gauge]

For $SU(2)$ and $SU(3)$, we find that the remaining field $\mathscr{X}$ is suppressed (exponential fall-off of the correlation function) in the low-energy or the long distance region.

4) We have given the numreical evidences for the dual Meissner effect caused by gauge-invariant magnetic monopoles in the Yang-Mills theory: simultaneous **formation of the chromoelectric flux tube** connecting a pair of quark and antiquark, and the magnetic current induced around the flux tube.

We have confirmed also the infrared restricted field dominance in the dual Meissner effect suggesting the magnetic monopole condensations.

5) We have determined the **type of the dual superconductivity** by measuring the penetration depth and the coherent length (assuming the relativistic Ginzburg-Landau model for fitting the data).

• For $SU(2)$, the type of the dual superconductivity is the **border between type I and II** or rather **weakly type I**.This is consistent with the preceding works.

• For $SU(3)$, the type is strictly **type I**.This is a new result which is consistent with the recent result of other groups.

These results support the **non-Abelian dual superconductivity of type I** as the mechanism of confinement of quarks in the fundamental representation in $SU(3)$ Yang-Mills theory.

Part III:
Unsolved problems in confinement

- derivation of area law of the Wilson loop average (proof of quark confinement)
- vortex condensation picture
- Casimir scaling of the string tension in the intermediate region
- N-ality dependence of the string tension in the asymptotic region
- dual (magnetic) symmetry, spontaneous symmetry breaking and dual Meissner effect

  $\vdots$

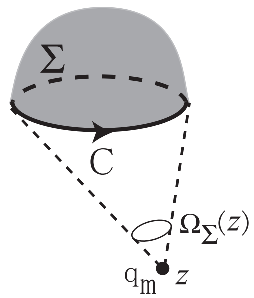
- deconfinement transition at finite temperature,
- chiral symmetry breaking/restoration and confinement/deconfinement crossover
- quark confinement, gluon confinement $\Longrightarrow$ Color confinement

§ Vortex vs magnetic monopole towards the area law

How the area law is derived in an analytical way?

In the dual superconductor picture, it is in principle achieved by summing up the contributions from the chromomagnetic monopoles to the Wilson loop average. However, it is not so easy to perform it in practice.

For $D = 3$, Ξ_Σ is the (normalized) solid angle Ω_Σ:



$$\Xi_\Sigma(x) = \Omega_\Sigma(x)/(4\pi). \tag{1}$$

For an ensemble of point-like magnetic charges located at $x = z_a$ $(a = 1, \cdots, n)$

$$k(x) = \sum_{a=1}^{n} q_m^a \delta^{(3)}(x - z_a), \quad q_m^a = 4\pi g_{\rm YM}^{-1} \ell_a, \quad \ell_a \in \mathbb{Z}, \tag{2}$$

the magnetic-monopole contribution to the $SU(2)$ Wilson loop operator reads

$$W_C^m = \exp\left[-ig_{\rm YM} J \int d^3x k(x) \Xi_\Sigma(x)\right] = \exp\left\{-ig_{\rm YM} J \sum_{a=1}^{n} \frac{q_m^a}{4\pi} \Omega_\Sigma(z_a)\right\} \tag{3}$$

we have a geometric representation:

$$W_C^m = \exp\left\{ iJ \sum_{a=1}^{n} \ell_a \Omega_\Sigma(z_a) \right\}, \quad \ell_a \in \mathbb{Z}. \tag{4}$$

The magnetic monopoles in the neighborhood of the Wilson surface Σ contribute to the Wilson loop

$$W_C^m = \prod_{a=1}^{n} \exp[\pm iJ(2\pi)\ell_a] = \begin{cases} \prod_{a=1}^{n}(-1)^{\ell_a} & (J = \frac{1}{2}, \frac{3}{2}, \cdots) \\ = 1 & (J = 1, 2, \cdots) \end{cases}. \tag{5}$$

Here, $\exp[-iJg_{\rm YM}(k, \Xi_\Sigma)]$ gives a non-trivial contribution:

$$\exp[\pm iJ(2\pi)] = (e^{\pm i\pi})^{2J} = (-1)^{2J} \in \mathbb{Z}(2) = \text{Center}(SU(2)), \tag{6}$$

for a half-integer or integer J from a magnetic monopole with a unit magnetic charge $q_m = 4\pi g_{\rm YM}^{-1}$ located on the minimal surface $\Omega_\Sigma(z) = \pm 2\pi$ for $z \in \Sigma$. This result does not depend on which surface bounding C is chosen in the non-Abelian Stokes theorem. [This helps us to understand the **N-ality** dependence of the asymptotic string tension.]

The vortex condensation picture gives an easy way to understand the area law.

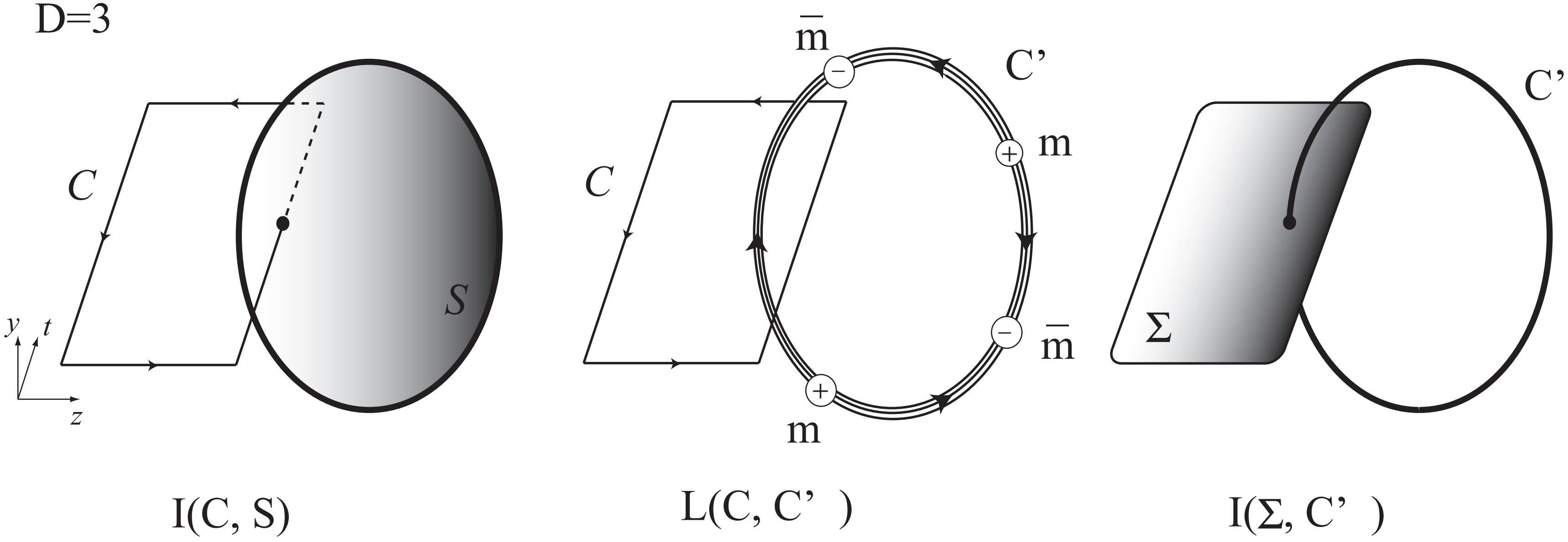
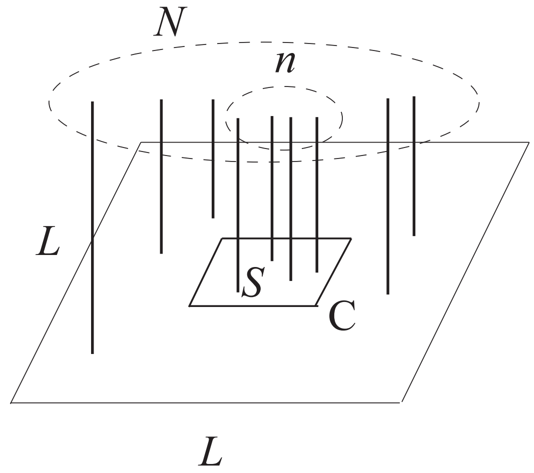


Figure 15: In $D = 3$ dimensional spacetime, it is assumed that the magnetic field emanating from a magnetic monopole m or an antimonopole $\bar{m}$ is squeezed into the flux tube to form the vortex line C'. The magnetic flux going along a vortex is just half of the total magnetic flux emanating from a monopole.

Let us assume that the vacuum is filled with percolating thin vortices. Suppose that N random vortices pierce a plane of area L^2. Each piercing multiplicatively contributes a factor $(-1)^{2J}$ to the Wilson loop average and n piercing within the loop take the value $(-1)^{2Jn}$. Then the probability that n of the piercings occur within an area S spanned by a Wilson loop is given by $(-1)^{2Jn}\left(\frac{S}{L^2}\right)^n (+1)^{N-n}\left(1-\frac{S}{L^2}\right)^{N-n}$.



Summing over all possibilities with the proper binomial weight yields

$$W_C = \sum_{n=0}^{N} \binom{N}{n} (-1)^{2Jn} \left(\frac{S}{L^2}\right)^n (+1)^{N-n} \left(1-\frac{S}{L^2}\right)^{N-n} = \left(1-\frac{S}{L^2}+(-1)^{2J}\frac{S}{L^2}\right)^N$$

$$= \left(1-\frac{[1-(-1)^{2J}]\rho S}{N}\right)^N \to \exp\{-\sigma_J S\} \quad (N\to\infty), \quad \rho := \frac{N}{L^2}, \tag{7}$$

where

$$\sigma_J = [1 - (-1)^{2J}]\rho = \begin{cases} \sigma_F = 2\rho & (J = \frac{1}{2}, \frac{3}{2}, \cdots) \\ 0 & (J = 1, 2, \cdots) \end{cases}, \tag{8}$$

where L has been eliminated in favor of the planar vortex density $\rho := N/L^2$. The limit of a large $N \to \infty$ is taken with a constant ρ. Thus one obtains an area law for the Wilson loop average with the string tension σ_J determined by the vortex density ρ. The crucial assumption in this argument is the independence of the piercing points. The asymptotic string tensions are zero for all integer-J representations (with N-ality or "biality" equal to 0), while they are nonzero and equal for all half-integer J representations (with N-ality equal to 1).

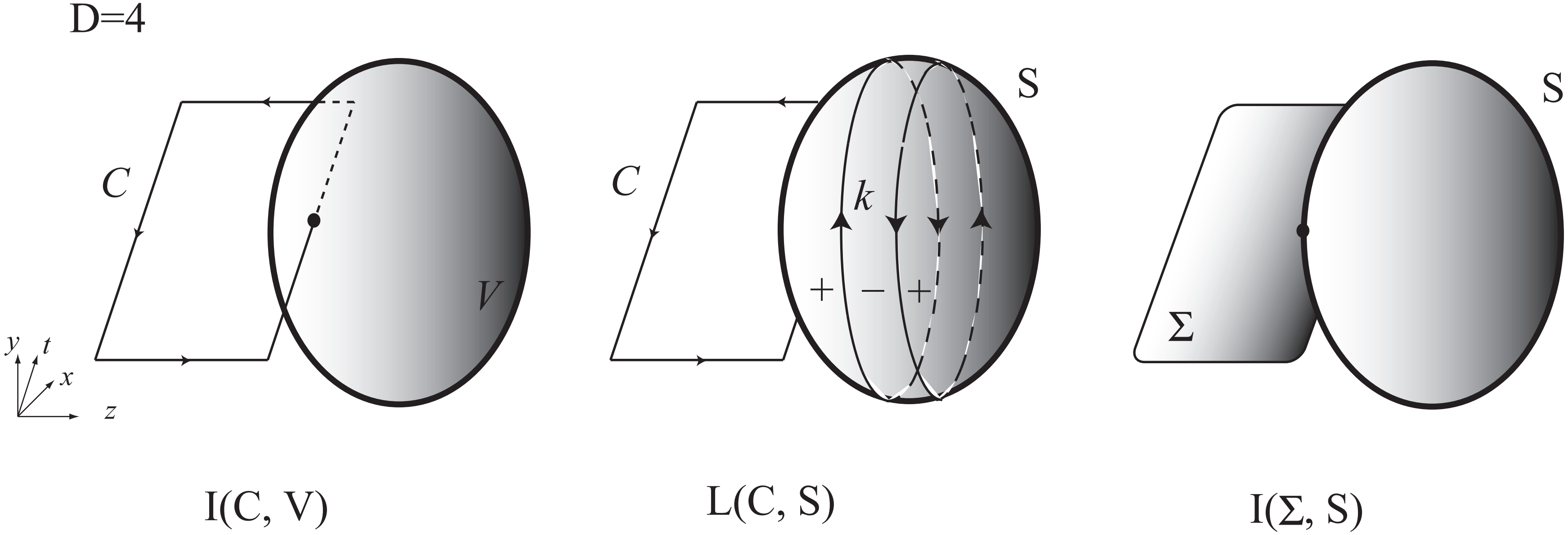


Figure 16: In $D = 4$ dimensional spacetime, it is assumed that the magnetic flux emanating from the magnetic-monopole current k is constrained on a surface to form the vortex surface S. A closed vortex surface S consists of a number of connected pieces S_n ($S = \cup_n S_n$), and each piece S_n has the magnetic-monopole current k at its boundary ∂S_n. This is possible if the closed vortex surface is non-orientable, although each piece of the vortex surface is orientable.

§ Casimir scaling and N-ality of the string tension

The static quark potential has the different behaviors depending on both the representation of the color group.

* Fundamental representation

$$V_F^{q\bar{q}}(r) \cong -\frac{\alpha_F}{r} + \sigma_F r + C, \tag{9}$$

* Adjoint representation

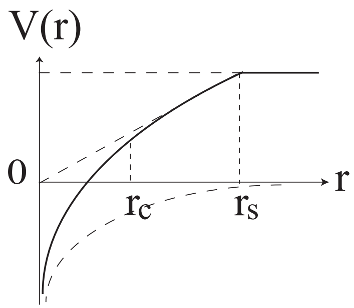


Figure 17: The $q\bar{q}$ potential $V(r)$ as a function of the distance r.

I:short distance region, Coulomb-like $V_A^{q\bar{q}}(r) \cong -\frac{\alpha_A}{r}$

II:intermediate region, a linear potential $V_A^{q\bar{q}}(r) \cong \sigma_A r$ with adjoint string tension σ_A
$\rightarrow$ Casimir scaling of the string tension

III:asymptotic region, a flat potential with the vanishing string tension $\sigma_A = 0$
$\rightarrow$ N-ality of the string tension

In the intermediate region according to numerical simulations, the static quark potential for quarks in the representation R of color group obeys the **Casimir scaling**:

$$V_R^{q\bar{q}}(r) \cong \frac{C_R}{C_F} V_F^{q\bar{q}}(r), \tag{10}$$

where C_R is the quadratic Casimir invariant of the representation R. Therefore,

$$\sigma_R \cong \frac{C_R}{C_F} \sigma_F. \tag{11}$$

$SU(2)$:$J = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ and $C_R = C_J = J(J+1)$

$$\sigma_J \cong \frac{C_J}{C_{1/2}} \sigma_{1/2} = \frac{4}{3} J(J+1) \sigma_{1/2}. \tag{12}$$

In particular, the adjoint string tension $\sigma_A = \sigma_{J=1}$ is related to the fundamental string tension $\sigma_F = \sigma_{J=1/2}$ as

$$C_F = C_{J=1/2} = \frac{3}{4}, \quad C_A = C_{J=1} = 2 \Longrightarrow \sigma_A \cong \frac{8}{3} \sigma_F = 2.66 \ldots \sigma_F. \tag{13}$$

$SU(3)$:

$$C_F = \frac{N^2-1}{2N}, \quad C_A = N \Longrightarrow \sigma_A \cong \frac{2N^2}{N^2-1}\sigma_F. \tag{14}$$

For the $SU(3)$ group, in particular, σ_A is related to σ_F as

$$\sigma_A \cong \frac{9}{4}\sigma_F = 2.25\sigma_F \quad (N=3), \tag{15}$$

while in the large N limit:

$$\sigma_A \to 2\sigma_F \quad (N \to \infty). \tag{16}$$

The Casimir scaling is supported by numerical simulations [Bali (2000)][Deldar (2000)]. The theoretical arguments are based on

- **factorization in the large N limit** [Greensite and Halpern (1983)]
- **dimensional reduction** to $D=2$ [Ambjorn, Olesen and Peterson (1984)]

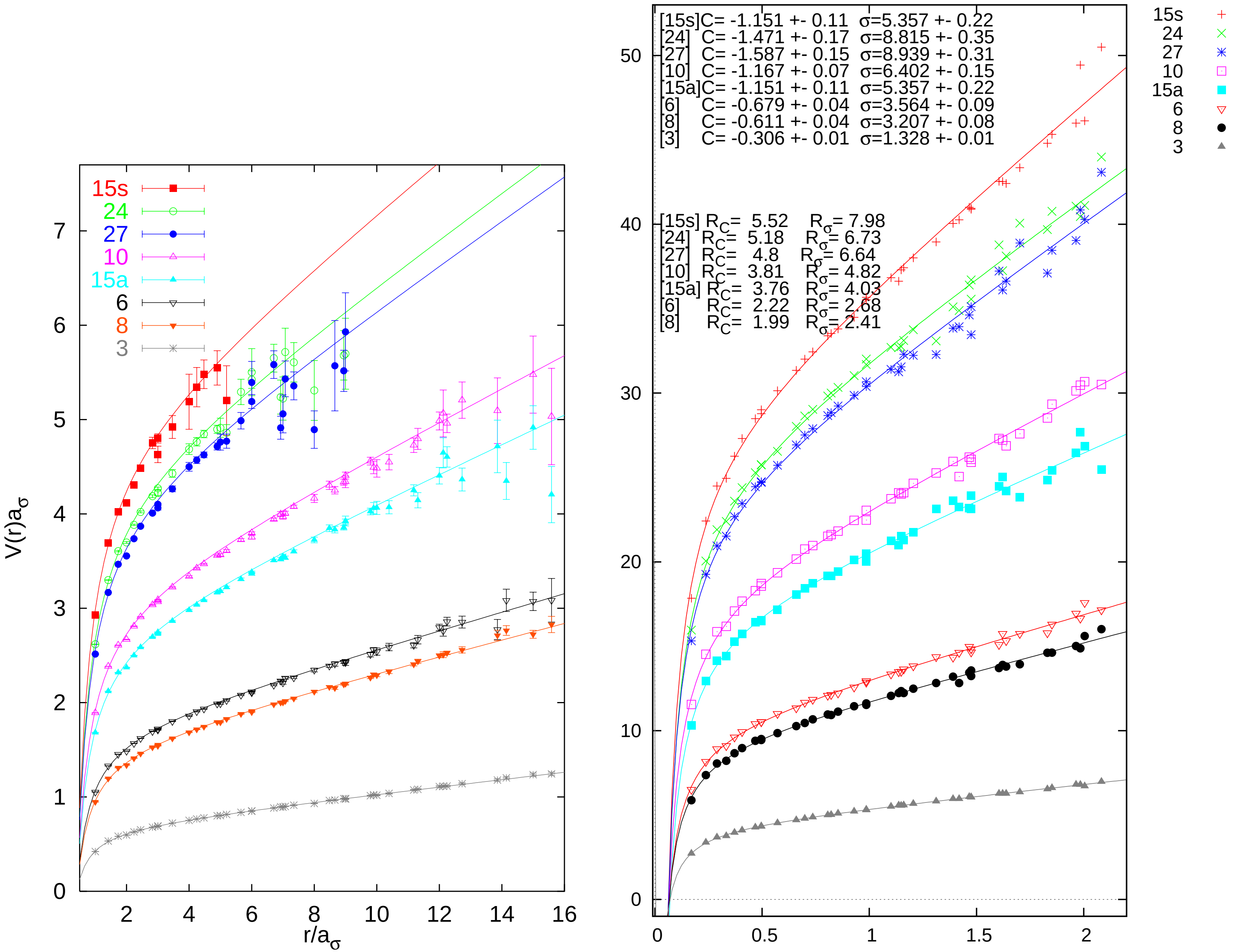


Figure 18: Static potentials between sources in various representations of $SU(3)$ in lattice units, $a_\sigma \approx 0.085$ fm. Casimir scaling due to the data of G.S. Bali, [hep-ph/0001312], Phys. Rept. **343**, 1–136 (2001).

The **asymptotic string tension** σ_R for quark in the representation R can only depend on the the N-**ality** k of the representation R, i.e., transformation properties of the representation for the quarks under the center of the gauge group:

$$\sigma_R = \sigma(k) := f(k)\sigma_F. \tag{17}$$

For $SU(N)$ group, the **center** elements consist of a set of N group elements proportional to the $N \times N$ unit matrix subject to the condition $\det(g) = 1$: $g = z_n := \exp(i\frac{2\pi}{N}n)\mathbf{1} \in Z(N)$ $(n = 0, 1, 2, ..., N-1)$. The center elements form a discrete Abelian subgroup known as $Z(N)$. The N-ality of a representation is equal to the number of boxes in the corresponding **Young tableau** (mod N). If the N-ality of a representation is k, then the transformation by $z \in Z(N)$ corresponds to multiplication by a factor $\exp(i\frac{2\pi}{N}nk)$.

However, the precise form of the k-dependence $f(k)$ is not known. There are two proposals. One is the "**Casimir scaling**" [Greensite(2003)]

$$\sigma(k) = \frac{k(N-k)}{N-1}\sigma_F. \tag{18}$$

Another is known as the "**Sine-Law scaling**" which is suggested by MQCD and softly broken $\mathcal{N}=2$ [Douglas and Shenker(1995)][Hanany, Strassler and Zaffaroni(1998)]

$$\sigma(k) = \frac{\sin\frac{\pi k}{N}}{\sin\frac{\pi}{N}}\sigma_F. \tag{19}$$

For the $SU(2)$ gauge group, the asymptotic string tension must satisfy

$$\sigma(k) = \begin{cases} \sigma_F & (k=1 : \text{J=half-integer}) \\ 0 & (k=0 : \text{J=integer}) \end{cases}. \tag{20}$$

For the $SU(3)$ gauge group, the two proposals give the same prediction:

$$\sigma(k) = \begin{cases} \sigma_F & (k=1, k=2) \\ 0 & (k=0) \end{cases}. \tag{21}$$

The two proposals give different predictions for $N \geq 4$. For fixed k, the Casimir scaling and the sine-law scaling are identical in the large N limit $N \to \infty$.

$$\sigma(k) = k\sigma_F. \tag{22}$$

The breaking of the adjoint string is difficult to observe in numerical simulations.

Thank you very much for your attention.